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RESISTANCE OF MATERIALS

WORKS OF

FRED B. SEELY, M.S.

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RESISTANCE OF MATERIALS

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BY

FRED B. SEELY, M.S.

Professor of Theoretical and Applied Mechanics
University of Illinois

SECOND EDITION

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PREFACE TO SECOND EDITION

In preparing the second edition of this book, careful consideration has been given to the suggestions offered by many students and teachers who have used the book during the past ten years.

The book has been completely reset. Many changes in wording, in arrangement, and in content have been made throughout the book. A large number of new problems has also been added.

The major changes consists: (1) in omitting Chapter III on elementary combined stresses, and distributing the material contained therein to other parts of the book where it has direct application in the treatment; (2) in placing the two chapters on the moment-area method of the treatment of beams in Part II where it may be omitted, if desired, without omitting pages in the assignment, or where it may be turned to in order to replace the double integration method if desired; (3) in placing the chapter on repeated loading in advance of the chapter on impact and energy loading, which makes it possible to better interpret the significance of localized stresses in relation to impact loading; (4) in replacing Part II on mechanical properties of structural materials by a brief treatment of a number of special topics in stress analysis that lie just beyond those usually included in the undergraduate courses in this subject; these topics may be assigned in courses which involve more time than is normally given to this subject, and they may also be helpful to the ambitious young graduate engineer who may wish to study further on his own initiative and direction.

It has been the aim in preparing the second edition to give special emphasis to the engineering significance of this subject, and not merely emphasis to mathematical methods of analysis.

A new feature of this edition consists of review questions and problems at the end of each chapter of Part I. They are designed to cover the essential part of the theory and applications; their purpose

is to direct the student in his review and to set before him a definite review assignment by means of which he may test himself.

The author wishes to express his appreciation of the interest of teachers and others in the first edition of the book, as shown by the many comments and helpful criticisms offered.

F. B. SEELY

URBANA, ILLINOIS
November, 1934

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RESISTANCE OF MATERIALS

PART I. ELEMENTARY TOPICS

CHAPTER I

STRESS AND STRAIN

1. Introduction.—Resistance of Materials is that branch of Mechanics which treats of the internal forces in a physical body and of the changes of shape and size of the body, particularly in their relation to the external forces that act on the body, and to the physical properties of the material of the body. The external forces that act on the body are called *loads*; the internal forces, which resist the external forces, are called *stresses*, and the accompanying changes in the dimensions of the body are called *deformations* or *strains*.

The *total stress* [1] on a section through a body is the total internal force acting on the section; that is, if the body is thought of as being divided in two parts by the section, the total stress at the section is the force exerted by one part of the body on the other part in order to hold in equilibrium the loads acting on the part considered. The component of the internal force normal to the area is called *normal stress*, and the component tangent to (or in) the area is called *shearing stress*. Further, a normal stress may be a tensile stress or a compressive stress according as the body is stretched or is shortened.

Intensity of stress or *unit-stress* [1] is defined to be stress per unit of area. In general the unit-stress varies from point to point over a section, its value at any point being considered to be the total stress on an elementary or differential part of the area, including the point, divided by the elementary area; but when the stress is distributed uniformly on an area the unit-stress at all points in the section is equal to the total stress on the area divided by the whole area. It is important to note that, in general, a unit-stress occurs *at a point* in a body *on some plane*

[1] In technical literature the term "stress" is sometimes used to denote what is here defined as intensity of stress or unit-stress, and the term "internal force" is used to denote what is here called total stress.

passing through the point, and that the unit-stress on the plane at the point considered may be a normal stress or a shearing stress.

A unit-stress may be expressed in various units, such as pounds per square inch (lb. per sq. in. or $lb/in.^2$), kilograms per square centimeter (kg. per sq. cm.), tons per square inch (ton per sq. in.), etc.

The *total deformation* or *total strain* in any direction is the total change in the dimension of the body in that direction, and the *unit-deformation* or *unit-strain* in any direction is the deformation or strain per unit of length in that direction.

The Problem Defined.—A body when subjected to loads is stressed and deformed, and the values of the stresses and the deformations in the body are of great importance in many engineering problems. These stresses are found by use of the general principles of mechanics (mainly of statics) and of the experimental laws that have been found to govern the action of the material. The main objects, then, in the study of Resistance of Materials are

(1) To determine the relation between the external forces (loads) acting on a body and the unit-stress at some point in the body on some plane passing through the point, so that the unit-stresses caused by known loads may be calculated. The point and plane selected will, in general, be the ones that give the stress which would first cause damage to the material if the loads were increased.

(2) To determine the relation between loads acting on a body and the resulting strains produced in the body so that strains may be determined from known loads, or vice versa.

(3) To obtain a knowledge of the physical properties, such as strength, stiffness, ductility, toughness, resilience, etc., of the various structural materials, since physical properties of the materials are involved in the relations under (1) and (2) and are also of importance in determining the suitability of a material for resisting loads under varying conditions of loading.

As suggested by the statements under (1) and (2) above, in some problems of design, the stress produced in the body by the load is the governing factor in the design, whereas in other problems the strain produced is the more important factor. For example, the chain of a hoist fulfills its function of lifting loads regardless of the amount of stretch of the chain that occurs, provided that the load does not produce a stress large enough to rupture the chain. That is, the stress developed in the chain by the maximum load to be applied is the governing factor

in the design of the chain. Likewise, stress is generally the governing factor in the design of many of the parts of bridges, buildings, cranes, ships, etc. In the design of machine tools, however, such as lathes, drill presses, grinding machines, etc., the deformations of the parts are frequently of prime importance since such machines will not produce work of sufficient accuracy if the deformations of the parts are too large.

2. Types of Loading.—With reference to the manner in which loads are applied or transmitted to a structure or machine, the loads will be considered under three distinct headings; namely, static loads, repeated loads, and impact and energy loads.

1. *Static, steady,* or *dead* loads are forces that are applied slowly and not repeated, and remain nearly constant after being applied to the body, or are repeated relatively few times, such as the loads on most buildings (a part of the load being the weights of the members of the structure), the load applied to a bar in a testing machine, etc.

2. *Repeated* loads are forces that are applied a very large number of times causing a stress (or stresses) in the material that is continually changing, usually through some definite range. For example, the loads applied to the connecting rod of an engine when the engine is running, the wheel loads on a railroad rail as a train passes over the rail, etc., are repeated loads. Repeated loads are discussed in Chapter X.

3. *Impact* loads are forces that are applied to the resisting body in a relatively short period of time; the shorter the time the greater the effect of the impact. An impact load, in general, is applied by a body that is in motion when it comes in contact with the resisting body, and the force exerted by the moving body and the period during which it acts can not in general be determined. For this reason in many problems it is more satisfactory to calculate the stress and strain produced by an impact load from the energy delivered to the resisting body by the moving body. When this is done the energy delivered to the resisting body is called an *energy load* and is expressed in foot-pounds (not in pounds). Impact and energy loads are considered in Chapter XI.

Other Classifications of Loads.—Loads may be classified as *distributed* loads and *concentrated* loads. A distributed load may be uniformly distributed or non-uniformly distributed. Thus, if sand is spread on a floor so that its depth is constant, the floor is subjected to a uniformly distributed load, whereas if the sand is distributed so that its depth is not constant the floor is said to carry a non-uniformly distributed load. A concentrated load is one whose area of contact with the resisting body is negligible in comparison with the area of the resisting body.

The more common types of loads to which structural and machine members are subjected may be classified as *central* loads, *torsional* loads,

and *bending* loads. A body may be subjected simultaneously to loads of any two or to all three of these types, however.

The stresses and strains caused by static central loads are discussed in this chapter. The stresses and strains caused by static torsional loads are considered in Chapter III, and those produced by static bending loads in Chapter IV.

STATIC CENTRAL LOADS

3. Stresses Due to Central Loads.—A central load is a concentrated load whose action line passes through the centroid of the area on which the stresses are to be considered, or a distributed load whose resultant passes through the centroid of the area. The stress produced by a central load may be any one of the three types, tensile stress, compressive stress, or shearing stress, but the distinguishing feature of a central

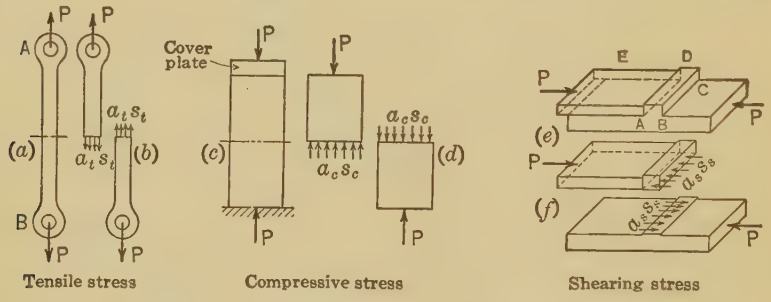

FIG. 1.—Tensile, compressive and shearing stresses.

load as compared to torsional and bending loads is that the stress may be assumed to be uniformly distributed over the area, that is, the intensity of stress (unit-stress) may be assumed to be constant. If a central load acts normal to the area it is called an *axial* load and it may be a tensile or a compressive axial load according as it produces tensile or compressive stress; if the central load lies in the plane of the area it is called a shearing central load. The compressive stress exerted by one body on another (different) body at the area of contact is frequently called a *bearing* stress.

Stresses due to central loads may be found as follows: Fig. 1(*a*) represents a straight bar AB subjected to an axial tensile load P causing tensile stress on any section of the bar perpendicular to the loads; Fig. 1(*c*) represents a bar in compression under the action of an axial load P causing compressive stress in the bar; and Fig. 1(*e*) represents a body under the action of a central shearing load P causing shearing stress on the area $ABCD$. The problem is to find the relation between the load

and the unit-stress developed at any point in the body on the plane specified, for each of these three types of central loads. Let a section be passed through the body in each case and a free-body diagram for either part of the body be drawn, as shown in Figs. 1(*b*), 1(*d*), and 1(*f*). Since the load in each case passes through the centroid of the area, the intensity of stress may be assumed to be constant; that is, the stress may be assumed to be uniformly [2] distributed over the area, and hence the total stress (called the resisting stress) is equal to the product of the area, *a*, and the unit-stress, *s*. Therefore, since each part of the body, in each case, is in equilibrium under the action of two collinear forces we may write

$$\text{load} = \text{resisting stress.}$$

Hence,

$$P = as, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which *s* may be a tensile, compressive or shearing unit-stress. Thus, in general, a central load, for any section of a body, is equal to the product of the area of that section and the unit-stress on the area.

It should be noted that the load *P* of Fig. 1(*e*) is not a central load unless the plate *E* is exceedingly thin, but in engineering computations shearing loads may frequently be assumed, without serious error, to be central loads even though they do not comply strictly with the above definition of a central load. Further, the loads *P* of Fig. 2 are axial loads for section *mn* but not for any other section, and if the loads were applied at points O_1 they would not be axial loads for any section of the body.

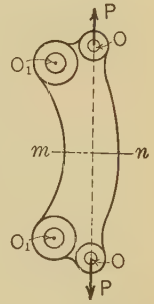

FIG. 2.—Central load with respect to one section only.

ILLUSTRATIVE PROBLEM

Problem 1.—In determining the strength of concrete, a test cylinder 8 in. in diameter and 16 in. high is loaded in a testing-machine as shown in Fig. 3(*a*). (*a*) If the maximum axial load *Q* that the concrete specimen can resist is 100,000 lb., determine the maximum unit-stress developed in the concrete. The dimensions shown in Fig. 3(*b*) and 3(*c*) were obtained from a testing-machine having a capacity of 100,000 lb.; when the concrete specimen is resisting its maximum load find (*b*) the maximum tensile unit-stress in each of the two screws of the testing-machine, (*c*) the shearing unit-stress in the threads of the screws, (*d*) the shearing unit-stress on the cylindrical area *AB* of the bronze bushing, and (*e*) the bearing unit-stress of the threads on the bushing.

Solution.—(*a*) Since the load *Q* on the concrete specimen is an axial load the

[2] For a discussion of the influence of abrupt changes in section of the body see Art. 95.

unit-stress on any section mn is uniformly distributed, and hence the total resisting stress is equal to $a_c s_c$ (see Fig. 1*d*). Hence,

$$Q = a_c s_c,$$

$$100{,}000 = \frac{\pi(8)^2}{4} \cdot s_c,$$

$$s_c = 1990 \text{ lb. per sq. in.}$$

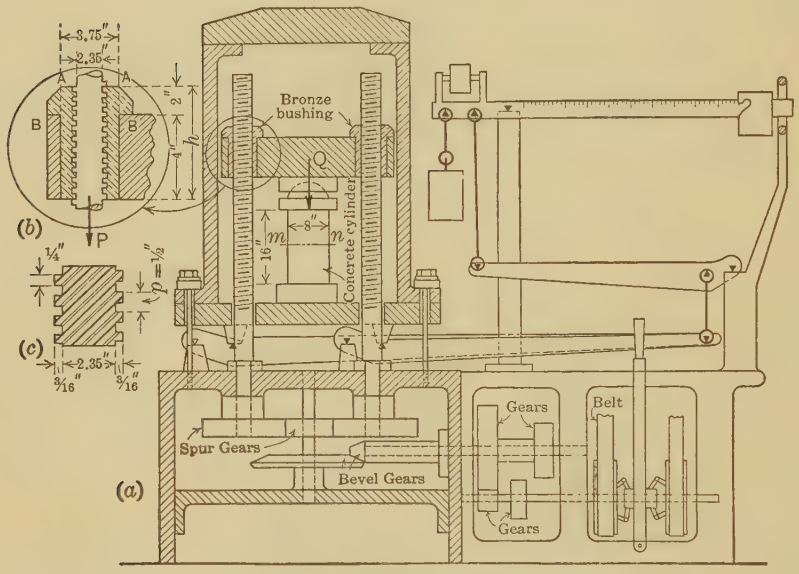

FIG. 3.—Testing machine.

Thus, the compressive stress in the concrete is 1990 lb. per sq. in.

(*b*) A tensile load P of 50,000 lb. is resisted by each screw. Hence,

$$P = a_t s_t,$$

$$50{,}000 = \frac{\pi(2.35)^2}{4} \cdot s_t,$$

$$s_t = 11{,}500 \text{ lb. per sq. in.}$$

The maximum tensile stress, then, that will be developed in the screws of this machine is 11,500 lb. per sq. in.

(*c*) The area of the thread on which the shearing stress is developed as the thread resists being stripped from the screw is,

Shearing area = a_s = thickness of thread × circumference at root of thread × number of threads, n, in the depth, h, of the bushing in which the screw turns $\left(n = \frac{h}{p},\right.$ where p is the pitch)

$$= \frac{1}{4} \times \pi \times 2.35 \times \frac{6}{0.5} = 22.15 \text{ sq. in.}$$

Now since the load of 50,000 lb. on each screw is resisted by the shearing stress on this 22.15 sq. in. of shearing area, we have,

$$P = a_s s_s,$$

$$s_s = \frac{50{,}000}{22.15} = 2260 \text{ lb. per sq. in.}$$

(*d*) The shearing area in the head of the bushing is

$$a_s = \text{circumference} \times \text{depth}$$

$$= \pi\, 3.75 \times 2 = 23.55 \text{ sq. in.}$$

The shearing unit-stress on this area due to the 50,000-lb. load is

$$s_s = \frac{P}{a_s} = \frac{50{,}000}{23.55} = 2120 \text{ lb. per sq. in.}$$

(*e*) The bearing area, a_b, of the threads on the bushing is

$$a_b = \tfrac{3}{16} \times \pi\, 2.54 \times 12 = 18.0 \text{ sq. in.}$$

The bearing unit-stress, then, is

$$s_b = \frac{P}{a_b} = \frac{50{,}000}{18.0} = 2780 \text{ lb. per sq. in.}$$

PROBLEMS

2. Two vertical posts are set 8 ft. apart and are connected at their tops by a flexible wire; the tops of the posts are on the same level. A weight of 100 lb. is suspended at the mid-point of the wire. When the wire is supporting this weight, its mid-point is 2 ft. below the level of the tops of the posts. If the diameter of the wire is $\frac{1}{8}$ in., what is the tensile unit-stress in the wire?

3. In the pin-connected truss shown in Fig. 4 each of the lower members, *AC* and *CE*, consists of two eye-bars (each eye-bar being similar to that shown in Fig. 1(*a*)). If the tensile unit-stress in each eye bar is 16,000 lb./in.^{2}, what is the cross-sectional area of each bar? *Ans.* 2.34 in.^{2}

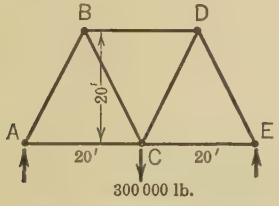

FIG. 4.—A pin-connected, pin-loaded truss.

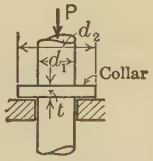

FIG. 5.—Collar bearing.

4. The bearing unit-stress for the collar bearing shown in Fig. 5 is 90 lb. per sq. in., and the compressive unit-stress in the shaft is 6000 lb. per sq. in. If the diameter, d_1, of the shaft is 4 in., what is the load, P, on the shaft and the diameter, d_2, of the collar? If the thickness, t, of the collar is $1\frac{1}{2}$ in., what is the shearing unit-stress on the area between the collar and shaft? Would it be desirable to use a multiple-collar bearing?

5. A timber frame shown in Fig. 6 carries a load, Q, of 12,000 lb. Find (a) the compressive unit-stress in the members A and B; (b) the horizontal shearing unit-stress in the timber C; and (c) the compressive (or bearing) unit-stress of C on the blocks D, which are 4-in. cubes.

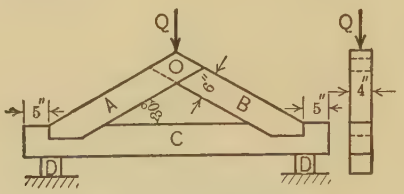

FIG. 6.—Timber frame.

6. If a specimen of wood is tested as shown in Fig. 7, and the maximum tensile unit-stress that the specimen can resist is 8000 lb. per sq. in., (a) what is the maximum axial load P that can be applied to the specimen? (b) What is the shearing unit-stress in the heads of the specimen when the load P is applied?

Ans. (a) 9000 lb.; (b) 750 lb./in.^{2}

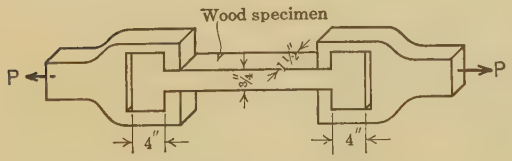

FIG. 7.—Tension and shear in wood specimen.

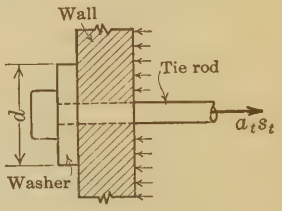

FIG. 8.—Bearing pressure of washer on wall.

7. A tie-rod 1 in. in diameter (Fig. 8) is used to help resist the lateral pressure against the walls of a bin. If the tensile unit-stress in the tie-rod is 15,000 lb. per sq. in., what diameter, d, should the washer have in order to keep the bearing unit-stress of the washer on the wall from exceeding 200 lb. per sq. in.?

Ans. 8.72 in.

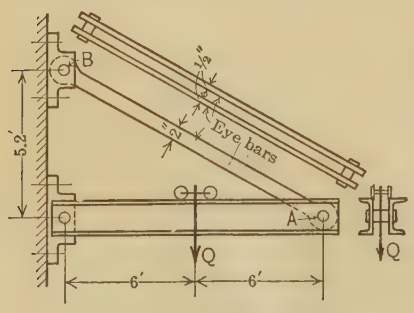

FIG. 9.—Pin-connected wall bracket.

8. The wall bracket shown in Fig. 9 carries a load, Q, of 10 tons. Find the tensile unit-stress in each of the two eye-bars, and the shearing unit-stress in the pins A and B if the diameter of each pin is $1\frac{1}{2}$ in.

Ans. s_t = 12,600 lb. per sq. in.
s_s = 7100 lb. per sq. in.

9. A pulley (Fig. 10) transmits a turning moment, $P \times d$, of 12,000 lb.-ft. to a 4-in. shaft, relative motion between the pulley and the shaft being prevented by a

flat key 1 in. wide, $\frac{5}{8}$ in. deep, and 6 in. long. Compute the shearing unit-stress in the key. *Ans.* s_s = 12,000 lb. per sq. in.

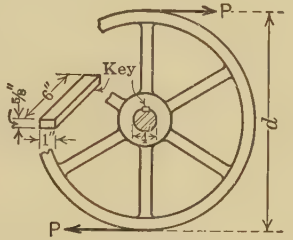

FIG. 10.—Shear in key connecting pulley and shaft.

10. A hollow gray cast-iron compression member supports loads as shown in Fig. 11. The member is prevented from bending by lateral supports (not shown). If the outside diameter is 20 in., what should be the inside diameter if the compressive stress developed is 12,000 lb./in.²?

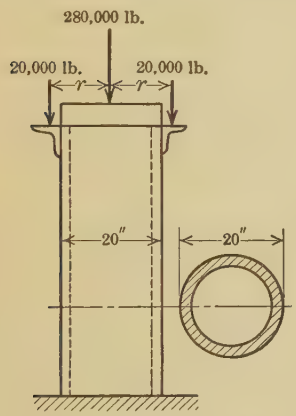

FIG. 11.—Hollow compression member.

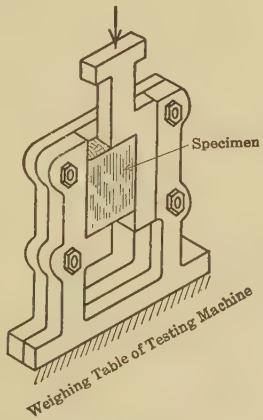

FIG. 12.—Shearing test of wood.

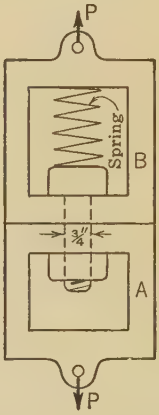

FIG. 13.—Bolt subjected to combination of loads.

11. In Fig. 12 is shown an apparatus for determining the shearing strength of wood. The depth of the specimen is 6 in. and its thickness is 1.5 in. If the load that causes the specimen to shear in two is 1200 lb., what is the maximum shearing strength of the wood?

12. In Fig. 13 the total tensile stress in the bolt is 3600 lb. The compressive force exerted on each end of the spring is 2000 lb., and the value of P is 1000 lb. Calculate the area of contact between the members A and B if the average compressive unit-stress on this area is 200 lb./in.² Neglect the strains of the members. *Ans.* 13 in.²

4. Stresses on Oblique Section.—As pointed out in the preceding article, if an axial load P is applied to a bar (Fig. 14*a*), the total stress, Q (Fig. 14*b*), on a cross-section perpendicular to the direction of the load is uniformly distributed on the area and hence is equal to the product of the area, a, of the section and the constant unit-stress, s. Further, since Q holds P in equilibrium it must be equal to, and collinear with, P and hence must pass through the centroid of the area a, that is, $P = Q = as$.

Now, if an oblique plane is passed through the centroid, O, of the area a and the lower part of the bar is removed (Fig. 14*c*), the total

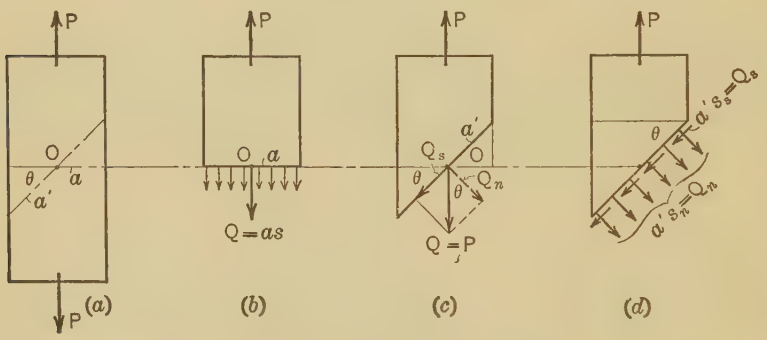

FIG. 14.—Stresses on oblique plane.

stress on the oblique area a' must likewise be Q in order to hold P in equilibrium. But Q is inclined to the area a' and, for convenience, will be resolved into normal (tensile) and tangential (shearing) components, Q_n and Q_s respectively (Fig. 14*c*), and since these components pass through the centroid of the area a' the normal and shearing intensities of stress, s_n and s_s on the area a' may be assumed to be constant (Fig. 14*d*). Hence, the following equations may be written:

$$Q_n = a's_n = \frac{a}{\cos\theta}s_n, \quad \text{and} \quad Q_s = a's_s = \frac{a}{\cos\theta}s_s.$$

But

$$Q_n = Q\cos\theta = P\cos\theta \quad \text{and} \quad Q_s = Q\sin\theta = P\sin\theta.$$

Therefore,

$$P\cos\theta = \frac{a}{\cos\theta}s_n \quad \text{and} \quad P\sin\theta = \frac{a}{\cos\theta}s_s$$

or

$$s_n = \frac{P}{a}\cos^2\theta \quad \text{and} \quad s_s = \frac{P}{a}\sin\theta\cos\theta = \frac{1}{2}\frac{P}{a}\sin 2\theta, \qquad (2)$$

from which the normal and shearing unit-stresses on any section inclined at an angle θ to the section on which the maximum normal stress occurs may be found. Now s_n will be a maximum when θ is zero, that is, on the plane perpendicular to P, and its value is $s = \frac{P}{a}$ which agrees with the value found in Art. 3. The maximum value of s_s in the above equation, however, occurs when θ is 45° and hence,

$$\text{maximum value of } s_s = \frac{1}{2}\frac{P}{a}. \quad . \quad . \quad (3)$$

Thus the maximum value of the shearing unit-stress in a bar subjected to an axial tensile (or compressive) load is equal to one-half the maximum normal stress developed in the bar, and it occurs on planes making angles of 45° with the plane on which the maximum normal stress occurs. But, since the shearing strength of many materials is much less than the tensile or compressive strength, the shearing unit-stress developed may be the more significant stress.

FIG. 15.—Stresses on 45-degree planes in tension member.

It will be observed from eq. (2) that the normal unit-stress s_n on the 45-degree plane is also equal to $\frac{1}{2}\frac{P}{a}$. Thus the unit-stresses on the 45-degree faces of a small block in a tensile member are as shown at B in Fig. 15.

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13. A brick when subjected to an axial compressive load fails in shear approximately on a 45-degree oblique plane, and hence its shearing strength is less than one-half of its compressive strength. If a brick 2 in. by 4 in. by 8 in. is tested in compression by applying axial loads to its 2 in. by 4 in. ends, what load is required to cause the brick to break if its shearing strength is 1000 lb/in.²?

14. A structural steel bar $\frac{3}{4}$ in. in diameter and 10 in. long is subjected to an axial tensile load of 12,000 lb. (*a*) Find the tensile and shearing unit-stresses on a plane making 60° with the direction of the load. (*b*) Find the maximum shearing unit-stress.

Ans. (*a*) s_n = 20,400 lb. per sq. in.; s_s = 11,700 lb. per sq. in. (*b*) s_s = 13,600 lb. per sq. in.

15. What will be the maximum shearing unit-stress developed in a steel bar 2 in. in diameter when subjected to an axial tensile load of 60,000 lb.?

16. A short concrete compression member having a square cross-section 8 in. on each side is subjected to an axial compressive load P. If the compressive unit-stress on any 45-degree plane through the member is 200 lb./in.², what value has the load P?

Ans. 25,600 lb.

5. Strains Due to Central Loads.—*Tensile or Compressive Strain.*—If an axial tensile load is applied to a straight bar of constant cross-section and of homogeneous material the bar is elongated or stretched and the strain per unit of length, that is, the unit-strain (unit-elongation) ϵ_t is given by the expression

$$\epsilon_t = \frac{e_t}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which e_t is the total tensile deformation and l is the original length of the bar. If the cross-section of the bar is not constant or if the material is not homogeneous, all unit lengths of the bar will not elongate the same amount, and the above expression then represents only the average unit-strain; thus, since the unit-strain varies from section to section along the bar, the unit-strain at any section of the bar is the ratio of the elongation de_t, of an elemental length dl, including the section, to the length dl. That is, the unit-strain at any section of the bar is

$$\epsilon_t = \frac{de_t}{dl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In order to determine ϵ_t from the above expression e_t, of course, must be expressed in terms of l (see the solution of Problem 20 where this is done).

If an axial compressive load is applied to the bar the above expressions for unit-strain also apply but the strain is a shortening. However, in loading a physical bar, it is practically impossible to obtain an axial compressive load; further, physical bodies are never homogeneous nor straight. Therefore, a compression member if relatively slender will bend when subjected to a load that is assumed to be axial. Bending action in compression members is discussed in Chapter VIII.

Shearing Strain.—Shearing strain (sometimes called detrusion) usually occurs in combination with tensile and compressive strain in connection with twisting and bending action, and it usually varies from point to point in the body. Although shearing strain due to a central shearing load cannot be found experimentally as can tensile and compressive deformations, the meaning of shearing strain and the quantitative measure of shearing unit-strain will be discussed at this point. Let Fig. 16(*a*) represent a bolt subjected to the shearing load P which tends to shear off the head of the bolt. The rectangle $ABCD$ of Fig. 16(*b*) represents a vertical section (enlarged) of that portion, $ABCD$, of the bolt that is subjected to shear. The head of the bolt exerts shearing stresses on the face BC and the lower part of the bolt exerts opposite

shearing stresses on the face AD. Other forces would have to act on $ABCD$ to hold it in equilibrium (that is P is not, strictly speaking, a *central* shearing load for section AD), but only the shearing forces will here be considered. The shearing forces shown in Fig. 16(b) cause the rectangle $ABCD$ to assume the form of a rhombus AB_1C_1D. The part $ABCD$ of the bolt subjected to shear may be considered to be made up of thin layers of materials each layer sliding a small amount relative to the layer beneath it.

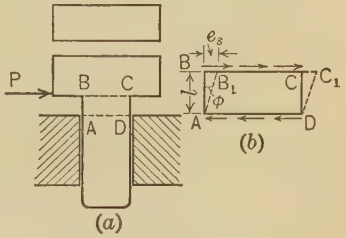

FIG. 16.—Shearing strain.

The total sliding, that is, the total shearing strain in the length AB or l is e_s, and the shearing deformation per unit length, that is, the shearing unit-strain ϵ_s, is

$$\epsilon_s = \frac{e_s}{l} = \tan \phi,$$

where ϕ is the change in the inclination of a line in the body that was originally perpendicular to the direction of the shearing deformation. Further, for a small angle, the tangent of the angle and the angle (expressed in radians) may be assumed to be equal without introducing serious errors. Therefore,

The shearing unit-strain at any point in a body is measured by the change in inclination (expressed in radians) of two lines that pass through the point and that were originally at right angles, one of the lines being parallel to the direction of the shear.

Hence

$$\epsilon_s = \frac{e_s}{l} = \tan \phi = \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It will be noted that a unit-strain (whether tensile, compressive, or shearing) is a length divided by a length and hence is an abstract number when each length is expressed in the same units; that is, its value is the same whether expressed as inches per inch or as feet per foot, etc.

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17. A straight bar 6 ft. long and $\frac{7}{8}$ in. in diameter is turned down to a diameter of $\frac{1}{2}$ in. for a distance of 2 ft. in its central portion. An axial load P causes a unit-deformation of 0.001 in the central 2 ft. and a total stretch of 0.04 in. in the whole bar. What is the unit-deformation of each of the end portions? *Ans.* 0.000333.

18. From compressive tests of concrete cylinders it has been found that the concrete fails when the unit-deformation is about 0.0012. How much does a specimen 8 in. in diameter by 16 in. high shorten before failure occurs?

19. In Fig. 16, if the shearing unit-strain at the point A is 0.001 and AB is 2 in. long, how far does the point B move to the right of A?

6. Stress-strain Curve. Proportional Limit. Yield-Point. Ultimate Strength. Elastic Limit. Modulus of Elasticity.—Experiments have shown that for nearly all structural materials the unit-stress in a material is approximately proportional to the accompanying unit-strain of the material provided that the unit-stress does not exceed a certain value. For example, let a steel bar (Fig. 17) of length l and of constant

FIG. 17.—Tensile test specimen.

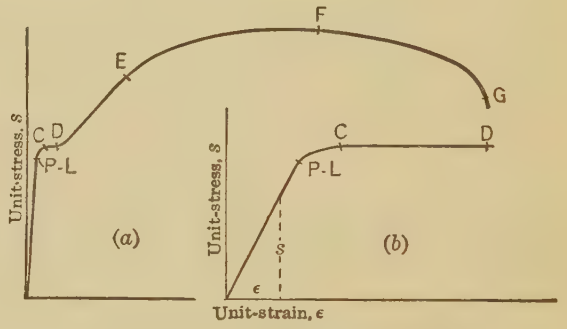


FIG. 18.—Stress-strain diagram for ductile steel.

cross-section, a, be subjected to an axial tensile load that gradually increases from zero value until the bar breaks. If P denotes any value of the load, then the unit-stress, s, corresponding to that load will be $\frac{P}{a}$. Further, let measurements of the stretch, e, of the bar be taken (by means of an extensometer) for various values of s; the unit-strain ϵ, then, may be found from the expression $\epsilon = \frac{e}{l}$. The relation between the unit-stress, s, and the unit-strain ϵ, found experimentally as indicated above, is represented, within quite approximate limits, by the *stress-strain* graph in Fig. 18(a), if the material is ductile such as low-carbon steel, wrought iron, etc., and by the stress-strain graph in Fig. 19 if the material is brittle,[3] such as high-carbon steel, etc.

[3] The stress-strain curves for concrete, cast iron, alloyed aluminum, bronze, etc., are curved practically all the way, but the first part of the curves, corresponding to relatively small stresses used in design, may be assumed without serious error to be straight lines.

Proportional Limit.—As indicated in Figs. 18 and 19, it is found that for most structural materials, as the unit-stress is increased the unit-strain is increased in practically the same ratio; if the unit-stress is doubled the unit-strain is likewise doubled, etc.; that is, the stress-strain curve is a close approximation to a straight line [3] until the unit-stress reaches a value called the proportional limit (or limit of proportionality). This unit-stress is represented on the stress-strain curve by the ordinate to the point *P–L* in Figs. 18 and 19. Therefore, the *proportional limit* of a material is defined to be the maximum unit-stress that can be developed in the material without causing the unit-strain to increase at a greater rate than the unit-stress increases, as load is applied to the body.

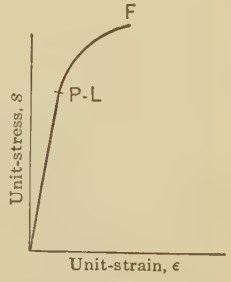


FIG. 19.—Stress-strain diagram for brittle material.

Yield Point.—As the load on the bar is increased further, causing a stress greater than the proportional limit, a unit-stress is reached at which the material continues to deform without an increase in load, provided that the material is *ductile.* The unit-stress at which this action occurs is called the yield-point, and is represented on the stress-strain curve by the ordinate to the part *CD* of the graph in Fig. 18 (only certain ductile metals have yield-points). Thus the yield-point of a material is defined to be the minimum unit-stress in the material at which the material deforms appreciably without an increase of load.

The stretch that occurs while the bar is stressed at its yield point is represented by the portion *CD* of the curve in Fig. 18. This stretch is large compared to the stretch that occurs in stressing the material to the proportional limit. Structural steel stretches at the yield-point as much as 0.025 in. per inch of length; thus a bar 10 in. long will stretch $\frac{1}{4}$ in. without an increase in load. Whereas, the unit-strain of a steel bar when stressed to its proportional limit is only about 0.0012; that is, a steel bar 10 in. long will have a total stretch at the proportional limit of $e = \epsilon l = 0.0012 \times 10 = 0.012$ in. In order to detect and measure the small strains that accompany stresses less than the proportional limit a strain-measuring apparatus (extensometer) is attached to the specimen; the load is applied and measured by means of a testing machine similar to that shown in Fig. 3.

Ultimate Strength.—If the load on the bar (Fig. 17) increases still further, the unit-stress and unit-deformation increase as indicated by the portion of the curve *DEF* (Fig. 18*a*) if the material is ductile, until the maximum unit-stress is reached, which is represented by the ordinate

to the curve at F and is called the ultimate strength. The ultimate strength for a brittle material is represented by the ordinate to F in Fig. 19; a brittle material breaks when stressed to the ultimate strength whereas a ductile material continues to stretch. The *ultimate strength* of a material, then, is defined to be the maximum unit-stress that can be developed in the material, as determined from the original cross-section of the bar or specimen; the cross-section of the bar decreases somewhat as the bar is stressed above the yield-point.

After the ultimate strength of a ductile material is developed, the bar begins to " neck down," thereby rapidly reducing the area of cross-section at the neck-down section (Fig. 20), and the load required to

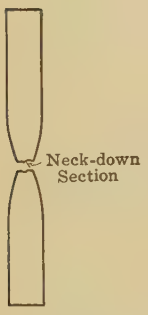


FIG. 20.—Form of ruptured specimen of ductile steel.

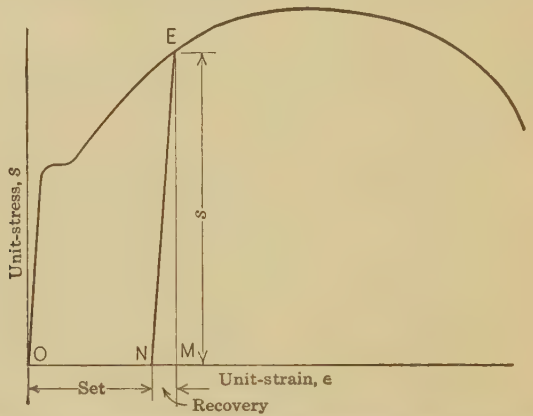


FIG. 21.—Stress-strain diagram showing permanent set.

cause the bar to continue to stretch decreases, as indicated by the curve FG (Fig. 18a). The load on the bar at the instant of rupture is called the breaking load. The breaking load divided by the area of the neck-down section is the value of the unit-stress in the bar when rupture occurs and this value is considerably greater than the ultimate strength. The ultimate strength, however, is of more importance than any stress in the bar after necking down has started, since the bar is in the process of failing after necking down starts.

Elastic Limit.—If the load on the bar is released after the bar has been stressed beyond the yield-point, the bar will not return to its original length, but will retain a part of its deformation. The deformation per unit of length retained by the bar after the load (and stress) has been reduced to zero is called the *permanent set* or merely *set.* For example, let the bar (Fig. 17) be stressed to the unit-stress, s, represented by the ordinate, $\overline{ME}$, to the point E on the stress-strain curve in Fig.

21, the unit-strain corresponding to the unit-stress s being $\overline{OM}$. If the load is gradually released the stress-strain curve will be represented by the line EN which is approximately parallel to the straight line portion below the proportional limit. That is, part of $\overline{OM}$, represented by $\overline{MN}$, is recovered, but a part, represented by $\overline{ON}$, is retained by the bar. $\overline{ON}$ therefore, represents the *set* corresponding to the unit-stress s.

If, however, the bar were subjected to a unit-stress equal to (or less than) the unit-stress called the elastic limit, the bar will return to its original length when the load (and stress) is reduced to zero. Hence, *the elastic limit* of a material is defined to be the maximum unit-stress that can be developed in the material without causing a permanent set.

In order to determine experimentally the value of the elastic limit for a material, a bar is subjected to a relatively small axial load and the load is released; if the extensometer shows that the bar has acquired no permanent deformation a larger load is applied and then released, etc. Thus, the stress in the bar is increased in small increments until a unit-stress is found which, on release of the stress, leaves a very small set in the bar, the minimum unit-stress at which set first occurs being the elastic limit. The results of tests show that for most structural metals the elastic limit of the metal has approximately the same numerical value as has the proportional limit of the material, and in technical literature, the proportional limit frequently is called (though incorrectly) the elastic limit.

Johnson's Proportional Limit.—The value of the proportional limit of a material may not be a reliable or satisfactory measure of the elastic strength of the material because the proportional limit is determined as the point of tangency of a straight line and a gradually sloping curve; this point is subject to considerable variation dependent upon the scale of plotting the stress-strain curve and other conditions not connected with the properties of the material. Moreover, a stress somewhat greater than the proportional limit may be set up in most structural material without destroying the usefulness of the material. With these facts in mind Professor J. B. Johnson proposed the following method of determining the so-called proportional limit (or apparent elastic limit as he called it).

A line OD (Fig. 22) is drawn on the stress-strain diagram which has a slope 50 per cent greater than that of the straight part of the curve below the proportional limit, and then a line $D'C'$ is drawn parallel to OD and tangent to the stress-strain curve; the ordinate to the point of tangency is the unit-stress at which the deformation is increasing, with respect to stress, 50 per cent faster than at any stress below the proportional limit.

The value of 50 per cent increase in slope is arbitrarily selected, the purpose being merely to select, by a convenient method, a unit-stress at which a small but measurable deviation from the straight stress-strain line which is indicative of the beginning of a general plastic yielding of the specimen.

Yield Strength.—For most load-resisting members any permanent deformation or set that accompanies stresses below the yield-point of the material does not seriously damage the members. The permanent deformation, however, that occurs at the yield point is sufficient to

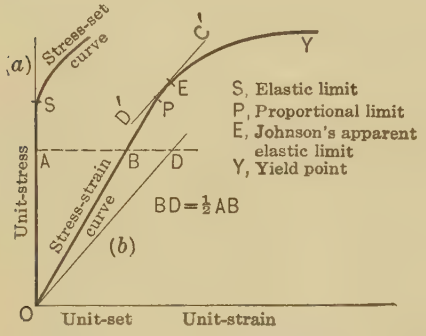


FIG. 22.—Methods of determining the maximum usable static strength of a ductile material.

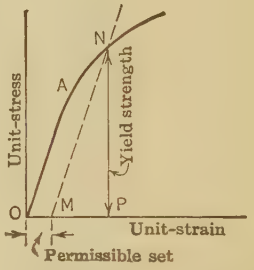


FIG. 23.—Method of determining yield strength.

destroy the usefulness of most members. The yield-point therefore is a good measure of the maximum useful strength of a material *provided that the material has a well-defined yield point.*[4]

Many metals (most of the non-ferrous metals and the high strength steels) do not possess yield points; for such materials a stress called the *yield strength* is sometimes used as a measure of the maximum useful strength of the material. It is defined as the maximum unit-stress that can be developed in the material without causing more than a specified permissible set; a permissible set of 0.10 or 0.20 per cent of the gage length is frequently used for metals. The value of the yield strength is found by first plotting a stress-strain curve (Fig. 23) from test data and then laying off on the axis of strain the permissible set *OM* and drawing a line from *M*

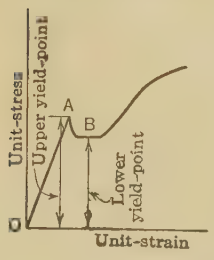


FIG. 24.

[4] Some of the more ductile metals have an upper and lower yield-point as indicated in Fig. 24. The lower yield point, however, as a rule indicates the maximum useful elastic strength of the material.

parallel to the straight-line portion OA of the stress-strain diagram; this line intersects the stress-strain curve at the point N; the ordinate PN is the yield strength.[5] This follows from the fact that if a specimen is stressed to a value corresponding to the point N and the stress is then reduced to zero, the stress-strain curve will follow closely the line NM, and the permanent set will be represented by OM. Hence NP is the unit-stress that is accompanied by the set OM which can be made any safe or permissible value and the corresponding yield strength PN found.

Both the yield strength and Johnson's proportional limit are satisfactory measures of the useful elastic strength of structural materials for most engineering uses.

Hooke's Law. Modulus of Elasticity.—If the bar discussed in the preceding article were subjected to a compressive stress, or to a shearing stress instead of a tensile stress, the first part of the stress-strain curve would be of the same form as that shown in Figs. 18 and 19. Hence, it follows that when a material is stressed *in one direction only*, the unit-stress at any point in a material is proportional to the unit-strain at that point, provided that the unit-stress does not exceed the proportional limit of the material. That is, for stresses within the proportional limit, all elastic material behaves according to the same law; namely, that the ratio of the unit-stress to the unit-deformation is a constant, or expressed mathematically,

$$\frac{s}{\epsilon} = \text{a constant}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

regardless of the kind of stress developed (whether tensile, compressive or shearing) and regardless of the way the stress and deformation are produced (whether by axial, bending or torsional loads, etc.). This is known as *Hooke's law.*[6] The numerical value of this constant ratio,

[5] This stress is sometimes, particularly in England, called proof stress.

[6] This law should be regarded only as statistically true as applied to actual physical bodies. For example, metals are composed of many crystalline grains with random orientations and with different physical properties, but any large group of such grains behaves approximately according to Hooke's law although the action in some individual crystals or grains may deviate markedly from this law. The deviation from the law in the case of large groups of crystals (bodies) involves strains so small, in comparison to the strains that may usually occur without harmful effect in most structural and machine members, that the deviations may be regarded as negligible. However, by use of very sensitive strain-measuring apparatus the relation between stress and strain is frequently found to be represented by a slightly curved line instead of a straight line, and in the design of certain members, notably springs, this deviation from Hooke's law may be of some importance. Furthermore,

however, is in general different for any one material when subjected to the different types of stress, and is also different for the different materials subjected to the same types of stress.

The numerical value of the constant [6] ratio of the unit-stress in a material to the accompanying unit-strain within the proportional limit is called the *modulus of elasticity of the material.* The symbol E will be used to denote the modulus of elasticity.

Thus we may write,

$$E = \frac{s}{\epsilon}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and, *when the stress and deformation are caused by axial loads,* the above expression may be written as follows:

$$E = \frac{\frac{P}{a}}{\frac{e}{l}} = \frac{Pl}{ae} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The value of the shearing modulus of elasticity can most easily be found when the shearing unit-stress and shearing unit-strain are produced by torsional loads as is discussed in Chapter III.

It should be noted that the modulus of elasticity is expressed in the same units as is unit-stress since ϵ is a ratio of length to length and is therefore merely a number.

As stated above, the values of the moduli of elasticity for any one material are, in general, not the same. For steel the tensile and compressive moduli of elasticity, E_t and E_c, are approximately 30,000,000 lb. per sq. in., but the shearing modulus, E_s, is approximately 12,000,000 lb. per sq. in. Similarly, any one of the moduli has different values for different materials. Thus, average values of the tensile moduli for steel, cast iron and timber (pine) are respectively 30,000,000, 15,000,000 and 1,500,000 lb. per sq. in.

It follows from the definition that the value of the modulus of elasticity of a material is represented by the slope of that portion of the stress-strain curve below the proportional limit; this portion of the curve is usually drawn to a large scale as in Fig. 18(*b*) so that the value

---

the value of E in $E = s/\epsilon$ for two specimens manufactured to the same specifications by different companies may vary as much as 10 per cent, although the variation usually is much less than this amount.

of the slope can be obtained with reasonable accuracy. Now this slope represents the rate at which unit-stress increases with unit-strain and hence the modulus of elasticity of a material is a measure of the stiffness of the material. That is, if one material has a modulus of elasticity twice as great as another material, the resisting stress in the one material, for a given strain, is twice as great as that in the other, and hence the one material is twice as stiff as the other.

Poisson's Ratio.—If a bar is subjected to an axial tensile load the bar is elongated in the direction of the load, that is, in the longitudinal direction, and at the same time the lateral dimension of the bar decreases. The ratio of the lateral unit-strain to the longitudinal unit-strain is called *Poisson's ratio*; it will be denoted by the symbol m. The value of this ratio for steel is approximately $\frac{1}{4}$; values of $\frac{1}{4}$ to $\frac{1}{3}$ are frequently used.

The shearing modulus of elasticity, E_s, of a homogeneous material may be expressed in terms of the tensile (or compressive) modulus of elasticity, E, and Poisson's ratio, m, by the following equation (see the author's Advanced Mechanics of Materials, p. 41),

$$E_s = \frac{E}{2(1 + m)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and since both E_s and E may be obtained from experimental data, this equation offers a convenient means of determining Poisson's ratio.

If the value of m for steel is taken to be $\frac{1}{4}$, then the above expression (for steel) becomes $E_s = \frac{2}{5}E$.

7. Properties of Structural Materials.—The proportional limit, elastic limit, yield-point, ultimate strength, modulus of elasticity, etc., of a material are usually called physical (or mechanical) properties of the material, the values of which are found from experimental results. Much of the material used in structures and machines is bought according to specifications which give the values of the physical properties that the material shall have for various uses. A number of technical societies and engineering companies have published specifications of structural materials; perhaps the most complete specifications are those of the American Society for Testing Materials called the "A.S.T.M. Standards."

Average values of various properties for a few of the more common engineering materials are given in Table 1. Attention is called to the fact that the properties of a commercial material necessarily vary somewhat due to the uncontrollable factors in the method of manufacture, treatment, etc., and that the specifications usually make reasonable

TABLE 1

AVERAGE VALUES OF STRENGTH, STIFFNESS, AND DUCTILITY OF VARIOUS STRUCTURAL MATERIALS

Material	Tensile Strength (lb. per sq. in.)		Compressive Strength (lb. per sq. in.)		Shearing Strength (lb. per sq. in.)		Modulus of Elasticity (lb. per sq. in.)		Elongation in 2 In. (per cent)
	Proportional Limit *	Ultimate Strength	Proportional Limit*	Ultimate Strength	Proportional Limit*	Ultimate Strength	Tensile and Compressive	Shearing	
Wrought iron	30,000	60,000	30,000	(*a*)	18,000	40,000	25,000,000	10,000,000	35
Structural steel (about 0.30% carbon)	35,000	65,000	35,000	(*a*)	21,000	45,000	30,000,000	12,000,000	30
Hard (high carbon) steel (about 0.60% carbon)	60,000	110,000	60,000	(*a*)	36,000	85,000	30,000,000	12,000,000	15
Structural nickel steel (3.5% nickel)	50,000	90,000	50,000	(*a*)	30,000	65,000	30,000,000	12,000,000	25
Chrome-nickel steel, heat treated { 0.40% carbon, 1.5 % nickel, 1.2 % chrome }	115,000	145,000	115,000	(*a*)	70,000	105,000	30,000,000	12,000,000	15
Ordinary gray cast iron	(*b*)	20,000	25,000	75,000	(*c*)	(*c*)	15,000,000	6,000,000	Negligible
Bronze, rolled (copper 95%, tin, 5%)	40,000	65,000	....	....	....	.....	14,000,000	6,000,000	30
Brass, rolled (copper 60%, zinc 40%)	(*b*)	60,000	....	....	....	.....	12,000,000	5,000,000	30
Aluminum alloy, rolled, tempered { Aluminum 96%, Copper 4% }	32,000	54,000	....	....	....	.....	10,000,000	3,800,000	15
Monel metal, alloyed nickel { Nickel 67%, Copper 28% }	40,000	75,000	....	....	....	.....	25,000,000	9,500,000	
Concrete, 1 : 2 : 4 mix (1 cement : 2 sand : 4 crushed stone)	....	200	....	2,500	....	.....	2,500,000		
Timber: Yellow pine, small clear specimens (*d*)	....	....	Parallel to grain 4,000 | 900 across grain 6,000			Parallel to grain = 800	1,500,000		
Timber: White oak, small clear specimens (*d*)	....	....	Parallel to grain 5,000 | 2,000 across grain 9,000		....	.....	2,000,000		

(*a*) The yield point, which is 2000 to 5000 lb. per sq. in. above the proportional limit, is the maximum usable compressive strength of a ductile material.

(*b*) Not well defined.

(*c*) Somewhat greater than the tensile strength since the material fails in tension when tested in shear.

(*d*) The strength of large-sized structural timber is only one-half of that of small specimens.

* This should be regarded as Johnson's proportional limit or as the yield strength, see Art. 6.

allowance for this variation by specifying a range of values or a minimum value for the properties which a material must have. For example, the ultimate strength of " Structural Steel for Building " as given in the " A.S.T.M. Standards " is 55,000 to 65,000 lb. per sq. in. and the yield-point must be not less than one-half of the ultimate strength.

The average values in Table 1 are presented with the view of helping the student to develop judgment in the use of engineering materials; the values in the table should be used when needed in the solution of subsequent problems. Appendix II contains tables giving values of the properties of many of the structural materials.

In commercial testing of ductile material, such as structural steel, the four quantities usually found are the yield-point, the ultimate strength, the elongation in per cent and the reduction of area in per cent; the proportional limit, elastic limit and modulus of elasticity are seldom found in commercial tests.

The percentage of elongation is found by dividing the increase in length of a bar after rupture has occurred by the original length and multiplying by 100; and the percentage of reduction of area is found by dividing the difference between the areas of the ruptured and original sections by the area of the original section and multiplying by 100. The percentage of elongation and of reduction of area are measures of the ductility of a material.

The following facts should be noted from a study of Table 1:

(*a*) The shearing proportional limit for steel is about six-tenths (0.6) of the tensile proportional limit.

(*b*) The ultimate strengths of the more ductile steels are about twice as large as the yield-points.

(*c*) The tensile moduli of elasticity of all grades of steel are equal, the value of the modulus being 30,000,000 lb. per sq. in.; thus the stiffness of steel is constant, whereas the strength of steel varies greatly with the composition and treatment.

(*d*) The shearing moduli of elasticity of all grades of steel are equal, the value of the modulus being two-fifths of the tensile modulus; $E_s = \frac{2}{5} E_t = \frac{2}{5}$ 30,000,000 = 12,000,000 lb. per sq. in.

(*e*) The usable ultimate compressive strength of *ductile* material is the yield-point of the material.

(*f*) The compressive strength of brittle material (cast iron, concrete, stone, etc.) is greater than the tensile strength.

(*g*) The strength of timber varies greatly with the direction of the grain.

ILLUSTRATIVE PROBLEM

Problem 20.—A bar is l in. long, has a constant cross-sectional area of a sq. in., and weighs w lb. per ft. of length per sq. in. of cross-section. Find the total elongation of the bar when it is suspended from one end and is subjected to no downward load except its own weight. If the bar is made of steel and is 400 ft. long, calculate the total stretch of the bar. (A steel bar having a cross-sectional area of 1 sq. in. weighs 3.4 lb. per ft. of length.)

FIG. 25.—Bar stretched by its own weight.

Solution.—The unit-stress, s_t, on a section at any distance y in. (Fig. 25) from the lower end expressed in lb. per sq. in. is

$$s_t = \frac{awy}{12a} = \frac{wy}{12}$$

and the unit-deformation at this section is

$$\epsilon_t = \frac{s_t}{E_t} = \frac{wy}{12E_t}.$$

The elongation of the short length, dy in., of the bar along which the unit-elongation may be assumed to be constant is

$$de_t = \epsilon_t dy,$$

and the total elongation (in inches) in the length of l in. is

$$e_t = \int_0^l \epsilon_t dy = \int_0^l \frac{wy}{12E_t} dy = \frac{w}{12E_t} \int_0^l y\, dy$$

$$= \frac{w}{12E_t} \frac{l^2}{2} = \frac{3.4 \times (400 \times 12)^2}{12 \times 30{,}000\,000 \times 2}$$

$$= 0.108 \text{ in.} = \text{elongation of steel bar 400 ft. long.}$$

PROBLEMS

21. If each eye-bar in Fig. 9 is stressed to 12,600 lb./in.^{2} by the load Q, how much does each bar stretch if the bars are made of steel? *Ans.* 0.066 in.

22. If the tie-rod in Fig. 8 is made of rolled brass and stretches 0.120 in. when stressed to 12,000 lb./in.^{2}, how long is the bar?

23. A hoisting cable $\frac{3}{8}$ in. in diameter supports a mine hopper that weighs 1000 lb. The length of the cable when carrying the hopper is 1650 ft., and the length of the cable when supporting the hopper and 1000 lb. of ore is 6.25 in. greater than when supporting the hopper alone. Find the modulus of elasticity of the cable.

Ans. $E = 28{,}600{,}000$ lb./in.^{2}

24. In Fig. 26 the load P acting on the pin-connected frame is 9000 lb. The tie-rod AC is made of steel. Calculate the stretch of the tie-rod assuming the cross-sectional area to be 0.25 in.^{2}

25. The boiler brace shown in Fig. 27 resists the pressure on an area of 80 sq. in. of the boiler-head. (*a*) If the steam pressure in the boiler is 120 lb. per sq. in., what is the tensile unit-stress in the brace? (*b*) If the brace is made of steel, how much does it elongate; assume that the rod is of constant diameter from pin to pin. (*c*) What is the shearing unit-stress in the three rivets at *A* if the diameter of each rivet is $\frac{3}{4}$ in.? *Ans.* (*a*) 17,200 lb. per sq. in., (*b*) $e = 0.0185$ in., (*c*) $s_s = 7240$ lb. per sq. in.

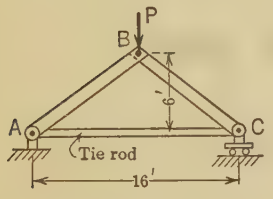


FIG. 26. — Pin-connected frame with steel tie-rod.

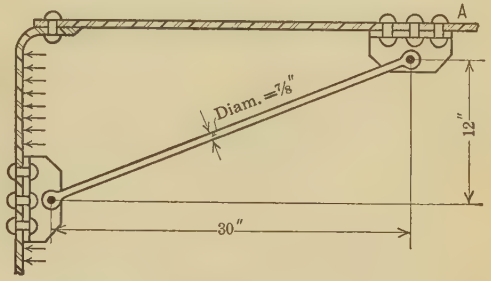


FIG. 27.—Brace in steam boiler.

26. It is specified that a steel rod 40 in. long is to be subjected to a unit-stress not greater than 10,000 lb. per sq. in. and to be elongated not more than 0.01 in. when resisting a tensile axial load of 20,000 lb. Determine the cross-sectional area required to satisfy each of the specifications and state which requirement governs the design. *Ans.* $a = 2$ sq. in. for strength, $a = 2.67$ sq. in. for stiffness.

27. Will a permanent set be caused in a wrought-iron bar $\frac{7}{8}$ in. in diameter and 3 ft. long when subjected to an axial tensile load of 24,000 lb.?

28. A steel piano wire with a constant cross-section of 0.00038 sq. in. has an elastic limit of 100,000 lb. per sq. in. If the wire is used to let down a 30-lb. body from the top of a building 500 ft. high, what must be the original length of the wire if the body just touches the ground when suspended by the wire? (Steel weighs 0.28 lb. per cu. in.)

29. In a tension test of a steel bar 0.499 in. in diameter the elongation was measured over a gage length of 2 in. Successive readings of the load and of the extensometer were as given herewith. Determine the elastic limit, the proportional limit, the yield-point and the modulus of elasticity of the material. Also find Johnson's proportional limit and the yield strength using a permissible set of 0.1 per cent of the gage length.

Load in Pounds	Reading of Extensometer in Inches
500	0.0
1500	0.0004
500	0.0
3000	0.0008
500	0.0
4490	0.0014
500	0.0
5980	0.0018
500	0.0
7510	0.0024
500	0.0
8630	0.0029
500	0.0002
9500	0.0075
500	0.004
9600	0.0130

8. Statically Indeterminate Axial-loaded Members.—If a body or simple structure is composed of two or more axially loaded parts or members, the equation of equilibrium, $P = as$, may not be sufficient to determine the stress in each part or member; that is, the stresses may be statically indeterminate. In such problems one or more additional equations involving the stresses in the members must be found. These additional equations are obtained from the relations between the deformations of the members; that is, from the geometry of the deformations or strains, and then using the relation between stress and strain, as is illustrated in the following problem.

ILLUSTRATIVE PROBLEM

Problem 30.—A short concrete compression member is reinforced with 12 rods of steel $\frac{1}{2}$ in. in diameter arranged in a circle with a 5-in. radius as shown in Fig. 28. A uniform pressure or load is applied to each end surface of the member. The concrete outside of the reinforcing rod is used for protecting the rods in case of fire and is not assumed to resist any of the load. If the load causes a compressive unit-stress of 600 lb. per sq. in. in the concrete, what is the unit-stress in the steel rods, and what is the total load carried by the member? The compressive moduli of elasticity of the concrete and steel are $E_1 =$ 2,000,000 lb. per sq. in., and $E_2 = 30{,}000{,}000$ lb. per sq. in., respectively.

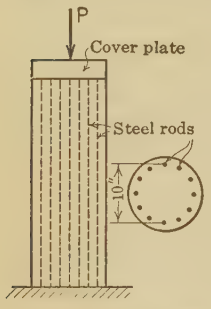


Fig. 28. — Concrete compression member reinforced with steel rods.

Solution.—Let the subscript 1 refer to concrete and the subscript 2 refer to the steel. From equilibrium we have

$$P = a_1 s_1 + a_2 s_2$$

but the unit-stress s_1 in the steel is unknown, and hence P cannot be found from this equation alone. But since the concrete and steel shorten the same amount, we may write

$$\epsilon_1 = \epsilon_2 = \frac{s_1}{E_1} = \frac{s_2}{E_2}; \text{ or, } \frac{s_2}{s_1} = \frac{E_2}{E_1} = 15$$

which means that the stress in the steel is 15 times as large as is the stress in the concrete. The stress in the steel is

$$s_2 = 600 \times 15 = 9000 \text{ lb./in.}^2$$

The total load P, then, is

$$P = \frac{\pi(10)^2}{4} \times 600 + \frac{\pi(\frac{1}{2})^2}{4} \times 12 \times 9000$$

$$= 68{,}300 \text{ lb.}$$

PROBLEMS

31. Two wires, one of steel for which E_t is found to be 14,500 tons per sq. in., and one of copper for which E_t is found to be 7500 tons per sq. in., have the same

length and carry equal axial loads. The copper wire has a diameter of 0.03 in If each wire elongates the same amount, what is the diameter of the steel wire?

Ans. $d = 0.0215$ in.

32. Two blocks, each 4 in. by 8 in. by 20 in., are bolted together to form a compression member as shown in Fig. 29. A pressure is applied to the top surface of the member causing both blocks to shorten the same amount. If one of the blocks is made of gray cast iron and the other of oak, find the total load P when the unit-stress in the cast iron is 25,000 lb. per sq. in. Also find how far from the center line of the block P must be applied.

33. A steel bolt extends through a cast-iron tube as shown in Fig. 30. What total stresses will be caused in the steel and the cast iron by one-fourth of a turn of the nut, if the length of the cast iron tube is 30 in., the pitch of the bolt threads is $\frac{1}{8}$ in., the diameter of the bolt (and inner diameter of the tube) is $\frac{3}{4}$ in., and the outer diameter of the tube is 1.5 in.? *Ans.* 8290 lb.

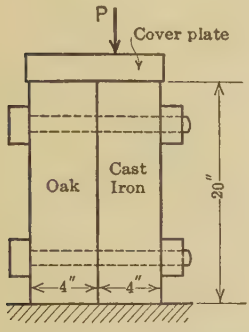


FIG. 29. — Two-material compression block.

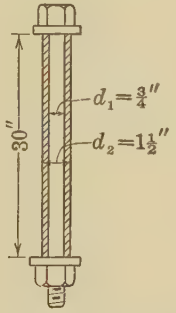


FIG. 30. — Steel bolt in cast-iron tube.

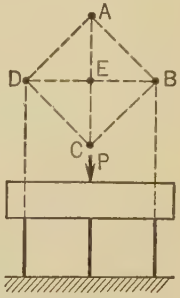


FIG. 31. — Four bars supporting rigid plate.

34. A thick rigid plate is supported on four bars as indicated in Fig. 31. The bars at A and C are steel bars and those at B and D are rolled bronze. Each bar is 2 in. in diameter and 20 in. long and is prevented from bending by lateral supports (not shown). The plate weighs 12,000 lb. What is the maximum load that can be applied at E without causing a stress greater than 10,000 lb. per sq. in. in the bronze bars?

9. Working Stress.—A *working stress* or an *allowable stress* for a material is the maximum unit-stress that is considered to be safe for the material when the material is resisting the loads that are assumed to be applied to it in service; it is the unit-stress used in the design of a member. The values of allowable stresses that are commonly used for various materials when resisting various types of loading have been determined largely from the results of tests of the materials and from the accumulated experience obtained in the construction and use of structures and machines under service conditions; Table 2 gives commonly used values of working stresses for several materials when the

material is subjected to static loads (see Table X of Appendix II for a more extensive table of values).

The allowable working stress in a member is always considerably less than the ultimate strength of the material of the member. For brittle materials (materials that do not have yield-points), such as concrete, cast iron, high carbon steel, etc., the working stress is frequently taken as a certain proportion of the ultimate strength of the material. For ductile materials, such as structural steel, wrought iron, etc., the working stress may be taken to be either a certain proportion of the ultimate strength or a certain (but different) proportion of the yield-point. In general, the working stress for a ductile material must be less than the yield-point since most structures would not fulfill their function if the members of the structure were stressed beyond their yield-points and thus became permanently deformed, although they might be safe against rupture or collapse.

Factor of Safety.—The term factor of safety has been widely used to denote the ratio of the ultimate strength of a material (or the yield-point) to the working stress used for the material. The use of this term and the idea of safety which it conveys, however, is likely to be misleading. For example, if a factor of safety of 4 based on the ultimate strength is used in selecting a working stress for structural steel, a structure designed on the basis of this working stress will *not* in general satisfactorily resist loads *four* times as great as those that produce the working stress, because the members in the structure will yield and take large permanent deformation before their ultimate strengths are reached and the distribution of the loads to the various members is thereby radically changed. The term *reduction factor* to provide a margin of strength is preferable to factor of safety.

Need of a Margin of Strength.—The need for selecting a working stress considerably less than the ultimate strength (or less than the yield-point) arises from (1) the uncertainties as to the properties of the materials used, (2) the uncertainties as to the loads to be resisted by the structure or machine as a whole and also by the various members of the structure, and (3) the uncertainties in the methods of calculating the stresses in the members. In deciding on the importance of these uncertainties and in selecting a value for the working stress many factors must be considered, such as the extent of the damage to property and life likely to be caused by a failure of the structure or machine, the degree of permanence or the length of useful life of the structure, the extent to which deterioration is likely to occur, the employment of periodic inspection, etc. The more the uncertainties are reduced the higher the working stresses may be. And, if the working stresses are increased,

the amount of materials purchased (and hence the cost of construction) will decrease.

TABLE 2

VALUES OF WORKING STRESSES FOR USE WITH STATIC LOADS
(See also Table X of Appendix II)
Values of stresses are expressed in lb. per sq. in.

Material	Type of Stress		
	Direct Tension	Direct Compression	Direct Shear
Structural steel	16,000–20,000	14,000–16,000	8000–10,000
Wrought iron	15,000	12,000	8000
Gray cast iron	3,000	15,000	.....
Timber (yellow pine)	.....	1000 with the grain 250 across the grain	.....
Portland cement concrete; 1 : 2 : 4 mix	.....	450	.....

In building construction, working stresses have become, to a large degree, standardized. Thus, the building laws of most large cities specify the maximum allowable working stresses to be used for the common structural materials (see Table X of Appendix II). For example, a working stress of 16,000 lb. per sq. in.[7] is specified by most building codes for tension members and for beams, etc., when made of rolled structural steel. In machine construction standardization of working stresses is more difficult than in structural work since the variety of materials used is greater and the conditions of service are more uncertain. However, a manufacturing firm that builds machines of certain types is able to standardize the working stresses for the design and construction of those machines.

The decrease in cost of construction accompanying the use of higher working stresses acts as a stimulus to reduce the uncertainties referred

[7] When structural steel is purchased according to standard specifications, and subjected to rigid inspection, working stresses higher than 16,000 lb. per sq. in. (18,000 to 20,000 lb. per sq. in.) are sometimes allowed in tension and flexural members; for steel *compression* members, however, the working stress is usually specified to be less than 16,000 lb. per sq. in. (usually from 12,000 to 14,000 lb. per sq. in.).

to above. Thus, the methods of manufacture and treatment of materials are being refined; tests and inspection of materials are becoming more rigid and more generally used; knowledge of the properties of materials and of the methods of calculating stresses in the members is becoming more extended and more generally applied, and experience and tests are gradually yielding more definite information as to the loads that various types of structures and machines are subjected to in service. The standardization of working stresses by law and by practice, of course, helps greatly to safeguard the public, who use engineering structures such as buildings, bridges, locomotives, elevators, etc., against unscrupulous or incompetent engineering design.

Factors Affecting Values of Working Stresses.—From Table 2 it will be noted that the working stresses for cast iron and timber are relatively small in relation to the ultimate strength of the material, the reason being that these materials are less uniform in structure than steel. For example, cast iron may contain blow holes, and initial stresses due to uneven cooling; timber may contain knots, pitch pockets, cracks, etc., and its strength is influenced markedly by its moisture content. Further, for a ductile material such as structural steel working stresses are relatively larger in relation to the ultimate strength than for a brittle material such as hard steel, cast iron, etc. The reason for this is that a ductile material deforms or yields if subjected to unexpected overloads or if high localized stresses occur at some portion of a member (and such stresses always do occur); the yielding distributes some of the excess stress to the surrounding materials and hence tends to prevent the member from breaking.

The working stress for any one material is, in general, smaller when the material resists impact and repeated loads than when it resists static loads; the reasons are that an impact load is in general less definitely known and it causes larger stresses and strains than does the same load when applied gradually, and localized stresses and nonhomogeneity of material have a much greater influence on the strength of the member when the loads are repeated than when applied but once as a static load.

PROBLEMS

35. Figure 32 represents a lower panel point of a pin-connected Pratt truss. The post and the three eye-bars are made of structural steel having an ultimate strength of 65,000 lb. per sq. in. and a yield-point equal to one-half the ultimate strength. Find the area of each of the eye-bars using a working stress equal to 0.4 of the yield-point. *Ans.* $a_P = 2.18$ sq. in., $a_Q = 3.08$ sq. in.

36. When a force, P, of 196 lb. is applied to the bell-crank shown in Fig. 33, the bearing pressure normal to the rubbing surfaces of the friction clutch is 15 lb. per

sq. in. If the coefficient of friction for the two surfaces is $\frac{1}{4}$, what is the maximum shearing unit-stress that can be developed in the rectangular key if its dimensions are $\frac{1}{2}$ in. wide, $\frac{5}{16}$ in. deep and $2\frac{1}{2}$ in. long? What is the factor of safety for this stress based on the shearing yield-point of the material if the key is made of structural steel? (Tensile yield-point = $\frac{1}{2}$ tensile ultimate strength; shearing yield-point = 0.6 tensile yield-point.) *Ans.* s_s = 2040 lb. per sq. in.; 9.6.

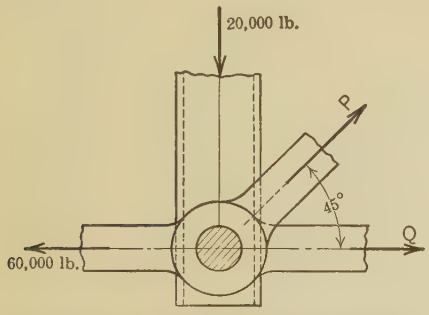


FIG. 32.—Stresses in pin-connected truss.

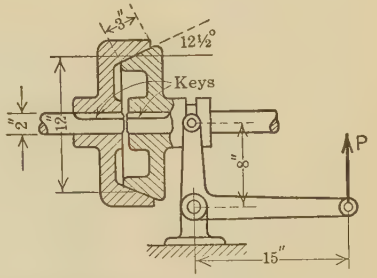


FIG. 33.—Shear in key of clutch.

37. The capacity of the hook and yoke shown in Fig. 34 is 10 tons. If the material has a yield-point of 40,000 lb./in.^{2} in tension and six-tenths as large in shear, what are the so-called factors of safety, based on the yield-points, of the tensile stress at section A and the shearing stress in the cylindrical pin when the hook is subjected to its maximum load? *Ans.* 4.15; 4.24.

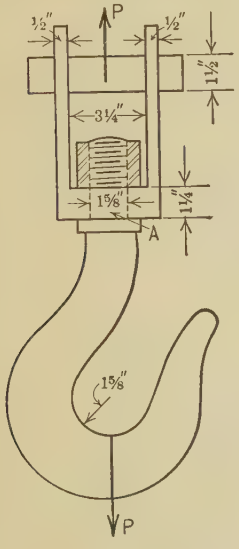


FIG. 34.—Crane hook.

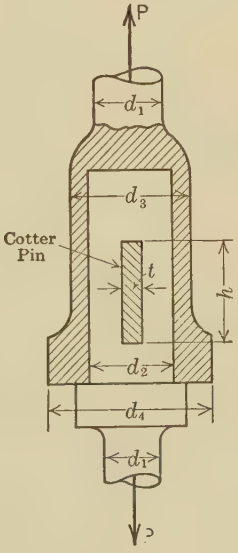


FIG. 35.—Socket joint and cotter pin.

38. Figure 35 represents two bars of diameter d_1 joined by means of a cotter pin, the axes of the bars being in the same straight line. The upper bar is enlarged at its

lower end to form a hollow cylindrical socket which fits over the enlarged upper end of the lower bar. The two bars and the cotter pin are made of structural steel. From the following data find the so-called factors of safety, based on the ultimate strength of the material, of the tensile stress in the two bars, the tensile stress in the socket, the shearing stress in the cotter pin and the bearing stress of the cotter pin on the socket. The ultimate bearing strength of the material may be assumed to be 80,000 lb. per sq. in.

$P = 60{,}000$ lb., $d_1 = 2$ in., $d_2 = 2\frac{1}{2}$ in., $d_3 = 3\frac{1}{2}$ in., $d_4 = 5\frac{1}{2}$ in., $t = 1$ in. and $h = 3$ in.

39. The maximum capacity P of the hydraulic jack used in the testing machine in Fig. 36 for determining the compressive strength of the specimen A is 100,000 lb. If the bars B are made of structural steel, find the diameter of each bar using a working stress having a so-called factor of safety of 5 based on the ultimate strength. If the screw C is made of 3.5 per cent nickel steel and has the same dimensions as those shown in Fig. 3, what should be the thickness h of the head if the shearing working stress in the threads is specified to be one-half of the shearing proportional limit? (See Table 1.)

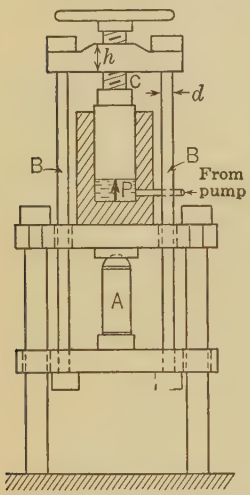


FIG. 36.—Method of loading in the Amsler testing machine.

40. Referring to Prob. 25 and Fig. 27 calculate the so-called factor of safety of the brace bar based on the yield-point of the material. The bar is made of wrought iron for which the tensile yield-point is one-half of the tensile ultimate strength.

41. If it is specified that the eye-bars in Fig. 9 shall not be subjected to a tensile stress in excess of one-fifth of their tensile ultimate strength, what is the maximum safe or working load Q that can be handled by this bracket crane; it should be noted that the eye-bars are stressed the greatest when the rolling load Q is at the end of the member (directly above A). The eye-bars are made of structural nickel (3.5 per cent Ni) steel.

42. In Fig. 37 A and B are steel pins 1 in. in diameter and C is a yellow pine timber member having a square cross-section 6 in. on a side. Calculate the dimension m using a working unit-stress with a factor of safety of 8 based on the shearing ultimate strength. Assume the grain of the timber is horizontal. *Ans.* $m = 3.33$ in.

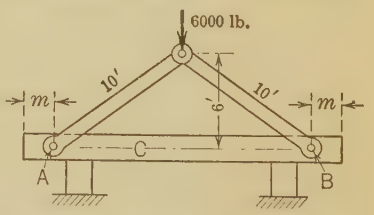


FIG. 37.—Shear in timber member.

10. Thermal Stresses.—When the temperature of a body is raised (or lowered) the material expands (or contracts) unless it is restrained from doing so by the forces exerted on it by other bodies. If Δt denotes the change in temperature of an unrestrained bar of material and n denotes the coefficient of thermal

expansion of the material (the change in length per unit of length per degree change in temperature), the total change in length of the bar of length l, obviously, is

$$\varepsilon = nl\Delta t; \quad \text{thus,} \quad \epsilon = \frac{e}{l} = n\Delta t, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which consistent units, of course, must be used. It is assumed that the change in temperature does not cause a change in the composition of the material.

Average values of n for several structural materials, using the Fahrenheit temperature scale, are as follows:

Brick	0.000 0050
Concrete	0.000 0062
Cast Iron	0.000 0062
Steel	0.000 0065
Brass	0.000 0092

If the bar at its ends is attached to fixed supports and its temperature is changed, the thermal expansion or contraction is prevented by the reactions at the ends which cause stress in the bar. This stress may be calculated by assuming that free thermal expansion or contraction first occurs and that the reactions are then applied to bring the bar to its original length. The change in length of the bar per unit of length accompanying the unit-stress s caused by the end reactions is

$\epsilon = \frac{s}{E}$, where E is the compressive or tensile modulus of elasticity.

Thus we may write

$$\epsilon = \frac{s}{E} = n\Delta t \quad \text{or} \quad s = En\Delta t, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in which consistent units must be used.

PROBLEMS

43. A steel tie-rod containing a turnbuckle has its ends attached to rigid walls and is tightened by the turnbuckle in summer when the temperature is 90° F. to give a stress of 2000 lb. per sq. in. What will be the stress in the rod in winter when its temperature is 20° F. below zero? *Ans.* 23,400 lb. per sq. in.

44. A thin brass ring when at a temperature of 200° F. just fits over a steel cylinder having a diameter of 4 in. and a temperature of 70° F. If the ring is left on the cylinder what will be the stress in the ring when it and the steel cylinder are cooled to 70° F., neglecting the deformation of the cylinder caused by the pressure of the ring.

45. A slab of 1 : 2 : 4 concrete (see Table 1) is 30 ft. long when at 20° F. How much longer will it be on a day in summer when its temperature is 120° F., if it is free to expand? If the slab is partially restrained from expanding by abutments at its ends so that only one-half of the above expansion is permitted, what is the compressive stress developed in the concrete? *Ans.* $e = 0.223$ in.; $s = 775$ lb./in.^{2}

46. A steel cable of a small suspension foot bridge is 200 ft. long on a hot day in summer (temperature = 100° F.). How much will it shorten when the temperature drops to − 40° F. in winter, if the stress in it does not change as its length decreases.

47. A thin steel hoop on a wooden tank is tightened at a temperature of 70° F. until the stress in the hoop is 4000 lb. per sq. in. The outside diameter of the tank is 6 ft. What will be the stress in the hoop when the temperature is − 10° F., assuming that no change occurs in the diameter of the tank. *Ans.* 19,600 lb./in.^{2}

48. A surveyor's steel measuring tape is exactly 100 ft. long at 70° F. when subjected to a 10-lb. pull. What correction should be made in the reading of the tape if the tape is used at 100° F. under a pull of 20 lb.? The tape is $\frac{1}{32}$ in. thick and $\frac{3}{8}$ in. wide.

11. Types or Phenomena of Failure and the Maximum Utilizable Strength of a Material.—The phenomena of failure of material considered in the preceding articles were limited to failures associated with static loads and ordinary temperatures. It is well to appreciate the phenomena of failure of material that occur under other important conditions of use.

A member is said to fail structurally (or to be structurally damaged) when, because of either excessive stress or strain, the member ceases to function properly in the structure or machine. For some materials and conditions of use structural damage does not occur until the member fractures (breaks); this is particularly true of brittle material subjected to static loads and of ductile material subjected to repeated loading. On the other hand, a member (particularly a member made of ductile material) may fail structurally because of general yielding (excessive permanent distortion of a considerable part of the member) although the character of this yielding and the maximum stress that can be developed in the material without causing excessive yielding may be very different at elevated temperatures from that at ordinary temperatures.

Several important types or phenomena of structural failure and the corresponding stresses that measure the maximum utilizable strength of the material may be described briefly as follows.

Yielding at Ordinary Temperatures.—A ductile material such as structural steel, when subjected to static loads at ordinary temperatures, fails structurally by *yielding of a considerable portion of the member*, characterized as *general yielding*. This yielding is accompanied by an

increase of strength (called strain-hardening), which at ordinary temperatures is permanent and prevents the yielding from continuing under the same load. The maximum utilizable strength of the material for the conditions of failure here considered is the yield-point if the material has a well defined yield-point, if not, the yield strength of the material is a satisfactory measure of the maximum utilizable strength.

Yielding (Creep) at Elevated Temperatures.—Most structural metals may be assumed to deform elastically at ordinary temperatures when stressed within the yield strength by static loads and they resist the loads without further deformation during the life of the structure or machine. These same metals, however, when stressed at elevated temperatures (above about 750° F. in the case of steel) such as exist in some pressure vessels, steam turbines, gas engines, chemical tanks, etc., continue to deform indefinitely under relatively low stresses, and eventually break. This *continuing* deformation is called *creep* or *flow*. A few of the softer metals such as lead creep at ordinary temperatures but the phenomenon of creep becomes of real importance at the elevated temperatures when a balance seems to be reached between the strengthening effect of strain hardening caused by the plastic yield and the weakening effect of the mobile or viscous condition produced by the higher temperature. Below a certain temperature the strain hardening apparently prevents appreciable creep and above this temperature the viscous condition of the material permits objectionable creep at stresses below the working stresses employed for the same material at ordinary temperatures.

The maximum utilizable strength of a metal under these conditions of use, therefore, is not the yield-point or the yield strength but the *creep strength* (or *creep limit*) which is defined as follows: The *creep strength* of a metal at a given elevated temperature is the maximum unit-stress that can be developed in the material during a specified length of time without causing more than a specified deformation; a time period of 10,000 or 100,000 hours and a deformation of 1 per cent of the gage length has been used rather widely, although other values are used depending on the service of the material involved. Values of creep strengths for steel at the upper range of elevated temperatures are much less than the yield strengths of the steel. For a discussion of a method of testing for determining the creep strength see Moore's Materials of Engineering (4th ed.) Chap. IV.

Sudden Fracture.—Some brittle materials under static loads deform so little before they fail that they perform their function satisfactorily at any stress up to their ultimate strengths; therefore, instead of failing by yielding they fail by fracturing. For such materials the maximum

utilizable strength is measured by the ultimate strength of the material. Likewise ductile materials sometimes fail by sudden fracture when subjected to impact loads, particularly when the member has an abrupt change of section such as a sharp groove at which the stress is highly concentrated.

Progressive Fracture under Repeated Loads.—As explained above, the yielding of a ductile metal member when subjected, at ordinary temperatures, to a static load that produces a calculated stress less than the yield-point of the material is not as a rule objectionable, and hence the member fulfills its load-resisting function satisfactorily. If, however, this same load is repeatedly applied, causing millions of applications of completely reversed stress, the member may fail by complete rupture or fracture without giving any evidence of yielding. A minute crack starts at one or more points in the member usually at points of high localized stress, such as at abrupt changes in section, and gradually spreads by continually rupturing the material at the edge of the crack where the stress is highly concentrated, until the area is greatly reduced and the member breaks suddenly. This process is usually called failure by *fatigue* but it is more accurately called *failure by progressive fracture.* (For further discussion of progressive fracture see Chap. X.)

The maximum utilizable strength of a material for resisting repeated stress is the *endurance limit.* This is defined as the maximum unit-stress that can be repeatedly developed in the material a very large number of times without causing the material to rupture. For values of endurance limits and for methods of testing to determine the endurance limit of a material see Chap. X.

REVIEW QUESTIONS AND PROBLEMS

49. Is the following statement correct? If a body is subjected to loads, the total stress on any cross-section that divides the body in two parts is the force exerted at the section on one part of the body by the other part in order to hold in equilibrium the external forces (loads) acting on the part considered.

50. Define unit-stress making clear that, in general, it occurs at a point in a body and on a plane passing through the point and that it may be either a normal or a shearing stress on that plane. Define unit-strain. The shearing unit-strain at a point in a strained body is measured by the tangent of an angle (or by an angle itself since the angle is small); define this angle.

51. Point out and correct the error in the following statement. If a straight bar of constant cross-section a is subjected to an axial load P, a shearing unit-stress s_s is developed on an oblique plane making an angle ϕ with the direction of the load such that $s_s = \frac{1}{2}\frac{P}{a}\cos 2\phi$. What is the maximum shearing unit-stress developed and on what plane does it act?

52. Select the correct word in each parenthesis in the following statement. The yield-point of a material is the (load, stress, unit-stress, unit-strain) in the material when the (unit-strain, area, stress) in the material increases without increase in the (length, unit-stress, energy, strain).

53. What load and area are used in calculating the ultimate tensile strength of a material from the data obtained in a test of a bar of the material?

54. What should be added to the following statement to make the definition complete? The modulus of elasticity of a material is the value of the ratio of the unit-stress in the material to the accompanying unit-strain in the material.

55. A bar of gray cast iron is subjected to a tensile unit-stress within the proportional limit of the material. If a structural steel bar having the same cross-section is subjected to the same unit-stress show, by use of the values in Table 1, that the cast-iron bar will stretch per unit-length twice as much as will the steel bar.

56. State three main reasons for selecting working stresses so that load-resisting members will possess a margin of strength.

57. Point out and correct the error in the following statement. The structural failure of a ductile steel member when subjected to a static load at ordinary temperature may be characterized as a yielding (permanent deformation of a considerable portion of the member, *not* accompanied by fracture) and hence either the ultimate static strength or the endurance limit of the material is a satisfactory measure of the maximum usable strength of the material for the conditions of use stated.

58. In Fig. 38 AB is a monel metal rod. If it is specified that the tensile unit-stress in the rod shall not exceed one-half of its proportional limit and that the rod shall not stretch more than 0.06 in., will a cross-sectional area of 0.5 in.^{2} for the rod satisfy the requirements? What should be the cross-sectional area of the pin at C if the pin is made of steel having a shearing yield-point of 24,000 lb./in.^{2} and the shearing stress in the pin has a factor of safety of 2 based on the yield-point.

Ans. $e = 0.048$ in. $a_s = 0.588$ in.^{2}

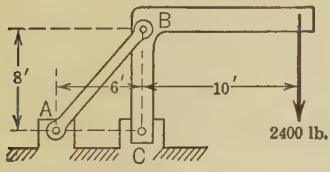


FIG. 38.

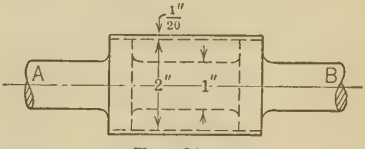


FIG. 39.

59. A brass tube is solidly attached to the enlarged portions of a steel shaft to form the body shown in Fig. 39. An axial tensile load P of 2900π lb. is applied at A and B. The tube and shaft between the enlarged portions stretch the same amount. The modulus of elasticity for brass is two-fifths that for steel. Calculate the tensile unit-stress in the tube and in the shaft between the enlarged portions.

Ans. 4000 lb./in.^{2} in tube; 10,000 lb./in.^{2} in shaft.

60. If the eye-bars in Fig. 9 stretch too much when made of structural steel, would this condition be improved by replacing the eye-bars with bars made of high-strength alloy steel such as chrome-nickel heat-treated steel (see Table 1), assuming the dimensions of the bars to be unchanged? Explain. *Ans.* No.

CHAPTER II

THIN-WALLED CYLINDERS AND SPHERES. RIVETED AND WELDED JOINTS

12. Stresses in Thin-walled Cylinders and Spheres.—A thin-walled cylinder or sphere is one in which the thickness of the wall or shell is small in comparison with the diameter of the vessel. When this condition is satisfied the stress in the shell due to an internal fluid pressure may be considered to be uniformly distributed on the longitudinal cross-sectional area of the shell without introducing serious errors in the calculation of the stress. Boilers, tanks, water and steam pipes, and hoops and tires that are shrunk on wheels usually may be treated as thin-walled cylinders. The cylinders of large guns and of some pipe, such as the pipes to hydraulic forging presses, etc., have thick walls, and the stress in the walls cannot be assumed to be uniformly distributed without introducing a large error in the calculated unit-stress.

The problem here considered is to determine the relation between the internal pressure in a closed thin-walled cylinder, the diameter of the cylinder, the thickness of the shell, and the intensity of stress in the shell (*a*) on a longitudinal section and (*b*) on a transverse section.

Stress on a Longitudinal Section.—Fig. 40(*a*) represents a portion of a thin-walled cylinder that is subjected to an internal fluid pressure of intensity R, the length of the portion being l. Let the diameter of the cylinder be denoted by D, the thickness of the shell by t, and the intensity of the tensile stress in the shell by s_t.

The pressure of the fluid on the internal surface of the cylinder at any point is normal to the surface at that point, as indicated in Fig. 40(*a*), and these internal pressures tend to rupture the cylinder on a longitudinal section (at AB and EF, Fig. 40*a*), the resultant pressure (or load) on one-half of the shell (Fig. 40*b*) being resisted, and held in equilibrium, by the total stresses P, P exerted by the other half of the shell at the areas AB and EF. Hence from equilibrium we have,

Resultant horizontal pressure or load = total resisting stress.

An expression for the resultant horizontal pressure on the half-shell may be found as follows: The resultant horizontal pressure on the semi-

cylindrical area, a, of the half-shell (Fig. 40*b*) is the sum of the horizontal components of the pressures on the elementary areas; the pressure on an elementary area da is Rda and its horizontal component is $Rda \cos \theta$, and hence the resultant horizontal pressure is equal to $\int Rda \cos \theta$, which may be written, $R \int da \cos \theta$ since the pressure R is the same at all points in the semi-cylindrical area, a. But $da \cos \theta$ is the area

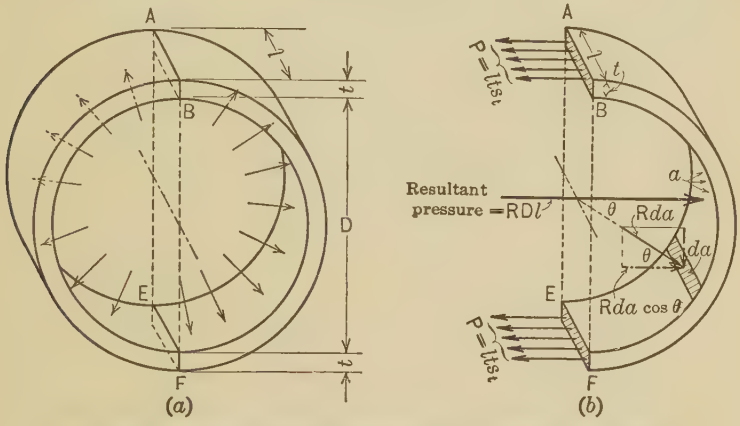


FIG. 40.—Stress on longitudinal section of thin-walled cylinder.

formed by projecting the area da on a vertical plane and hence $\int da \cos \theta$ is the area formed by projecting the semi-cylindrical area on a vertical plane, and is, therefore, equal to Dl. Thus, the resultant horizontal pressure is RDl.

Further, since the shell is thin, the resisting stress, P, may be assumed to be distributed uniformly over each of the two areas, and hence $2P = 2a_t s_t = 2lts_t$. Therefore, the above equation becomes

$$RDl = 2lts_t.$$

Whence,

$$s_t = \frac{RD}{2t}. \quad \ldots \ldots \ldots \quad (13)$$

Stress in Transverse Section.—The total pressure of the fluid against the end of the cylinder must be resisted by the total stress on a transverse section of the cylinder as indicated in Fig. 41. Now the total pressure against the end of the cylinder is $R\frac{\pi D^2}{4}$, and the total

resisting stress is $\pi Dt \cdot s_t$. Hence from the condition of equilibrium we have,

$$R\frac{\pi D^2}{4} = \pi Dt\ s_t,$$

whence,

$$s_t = \frac{RD}{4t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In (13) and (14) R and s_t must be expressed in the same units (usually pounds per square inch), and D and t must be expressed in the same units (usually inches).

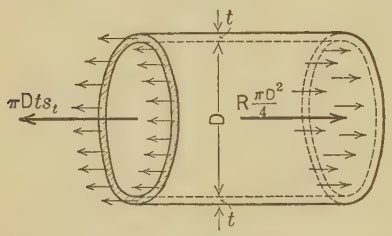


FIG. 41.—Stress on transverse section of thin-walled cylinder.

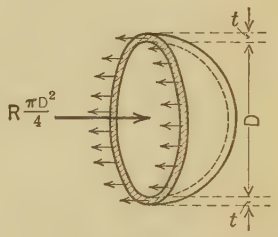


FIG. 42.—Stress in thin-walled sphere.

A comparison of equations (13) and (14) shows that the intensity of stress on a longitudinal section of a thin-walled cylinder due to an internal fluid pressure is twice as great as that on a transverse section of the same cylinder. This fact explains why the riveted joints connecting the plates of a boiler along a longitudinal seam requires more rivets than that along a transverse seam.

Equation (14) may be used also to find the unit-stress in a thin-walled sphere due to internal fluid pressure as will be evident when the method used above in determining the stress in a thin-walled cylinder is applied to a thin-walled sphere as represented in Fig. 42.

FIG. 43. — One-half of cylindrical shell and fluid contained therein.

PROBLEMS

61. Derive eq. (13) by considering the forces acting on one-half of the cylindrical vessel and the fluid contained in that half of the vessel as suggested in Fig. 43, instead of considering the forces acting on the half-shell alone as is done in Fig. 40.

62. A boiler is subjected to an internal pressure of 160 lb. per sq. in. and is 48 in. in diameter. (*a*) What is the total stress acting on (or transmitted across) each of the two longitudinal cross-sectional areas of the plate or shell in a length of 3 in. along the boiler? (*b*) What is the unit-stress developed in the shell if the thickness of the shell is $\frac{3}{8}$ in.?

63. A standpipe 6 ft. in diameter is 60 ft. high. When it is full of water, what is the circumferential unit-stress in the plate at the bottom of the standpipe if the thickness of the plate is $\frac{3}{8}$ in.? Hint: Since water weighs 62.5 lb. per cu. ft., the pressure (in all directions) at the bottom of the standpipe is 62.5×60 lb. per sq. ft.

Ans. $s = 2500$ lb. per sq. in.

64. The pressure in the cylinder of a steam-engine (Fig. 44) is 120 lb. per sq. in. and the internal diameter, D, of the cylinder is 14 in. How many $\frac{3}{4}$-in. bolts are required for strength if the tensile unit-stress in the bolts is not to exceed 8000 lb. per sq. in.? What should be the thickness, t, of the walls of the cast-iron cylinder to satisfy the requirement for strength if the allowable tensile stress is 800 lb. per sq. in.? NOTE.—The allowable stresses are taken low due to the fact that the load is applied with more or less impact. Further, the requirement for strength in many problems is not the governing requirement. In this problem, for example, the bolts should be large enough to prevent a workman, with ordinary tools, from twisting the heads off. Further, the requirement for tightness of the joint may determine the number of bolts. Similarly, the thickness of the wall may be influenced by the considerations of heat loss or of ease and reliability in casting, etc.

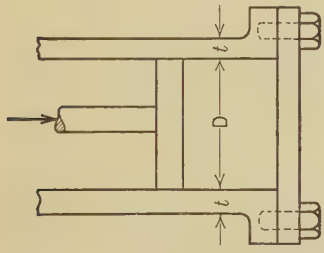


FIG. 44.—Steam cylinder and piston.

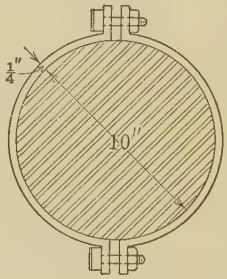


FIG. 45.—Band tightened on smooth cylinder.

65. An aluminum band $\frac{1}{4}$ in. thick and 2 in. wide is tightened on a smooth rigid steel cylinder (Fig. 45) by turning the nuts on the bolts until the tensile stress in each of the two bolts is 10,000 lb. (*a*) What is the pressure per unit-area of the cylinder on the band? (*b*) How much does each half band stretch? (*c*) What is the unit-stress in the band? *Ans.* (*a*) 1000 lb./in.^{2}; (*b*) 0.0314 in.; (*c*) 20,000 lb./in.^{2}

66. Water pipes are frequently made of cast iron. According to specifications a pipe 18 in. in diameter must have a wall thickness of 0.87 in. and must resist an internal pressure of 300 lb. per sq. in. What circumferential unit-stress is developed in the pipe? *Ans.* $s = 3100$ lb. per sq. in.

67. The steel tire for a locomotive driving-wheel has an internal diameter $\frac{1}{1500} \cdot d$ less than that of the wheel on which the tire is to be shrunk, where d is the diameter of the wheel. The value of d is 60 in. and the value of t, the thickness of the tire, is $\frac{3}{4}$ in. If it is assumed that after the tire is shrunk on the wheel the diameter of the wheel is not changed by the pressure of the tire, find (*a*) the elongation of the tire, (*b*) the tensile unit-stress (hoop-tension) in the tire, and (*c*) the intensity of the pressure of the tire on the wheel.

Ans. (*a*) $e = 0.126$ in.; (*b*) $s = 20{,}000$ lb. per sq. in.; (*c*) $R = 500$ lb. per sq. in.

13. Riveted Joints.—Riveted joints are important structural elements in buildings, bridges, cranes, etc., and also in pressure vessels such as boilers, tanks, water pipes, etc.

The load causing the stresses in the various parts of a riveted joint (shearing stress in the rivets, bearing stress on the rivets, tensile stress in the plate, see Fig. 48) is usually assumed to act through the centroid of the area on which the stresses occur, and hence the stress on the resisting area is assumed to be uniformly distributed. This assumption, however, must be regarded as a rough approximation only, since, as discussed in Art. 17, the load on a riveted joint is seldom distributed evenly to the rivets; however, the design of the simpler types of joints is usually based on this assumption. The stresses in riveted joints produced by eccentric loads are discussed in Chapter VII, but eccentric loads, whenever possible, should be avoided.

According to the above assumption the stresses in a riveted joint may be found from the equation developed in Chapter I for a central load; namely, $P = as$, in which P is the central load transmitted from one to the other of two plates that are connected by the rivets, and s is any one of the various unit-stresses (shearing, bearing, tension, etc.) that may be the cause of the failure of the joint (as discussed in Art. 15) and a is the area on which the stress is distributed.

14. Types of Riveted Joints. Definitions.—It is convenient to divide riveted joints into two general groups; (1) structural joints used in connecting members in bridges, buildings, cranes, and other structures, and (2) boiler, tank, or pipe joints used in connecting plates in various types of pressure vessels. In structural joints, strength is the main requirement, whereas in boiler and pipe joints tightness in addition to strength must be considered.

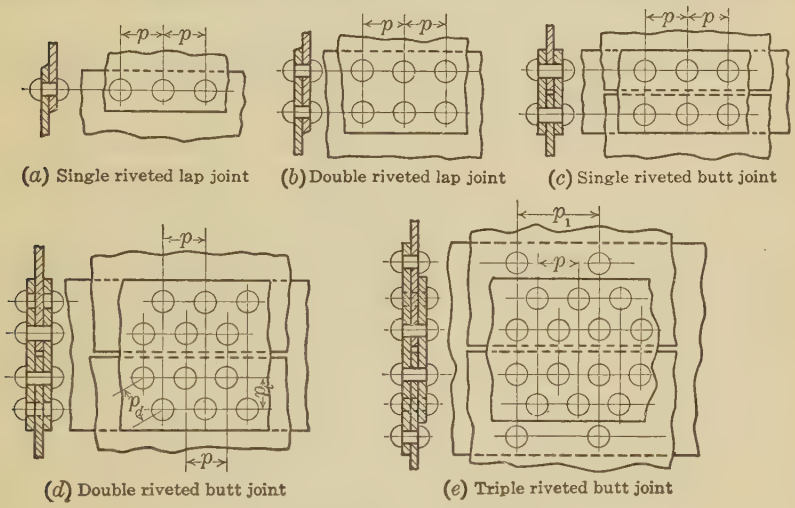
Two types of joints are widely used for both the groups mentioned above; namely, lap joints and butt joints (see Fig. 46), and both lap and butt joints may be single riveted, double riveted, triple riveted, etc., according as one, two, three, etc., rows of rivets pierce each of the two plates that are connected. Further, a butt joint may have two cover plates (or straps) or only one cover plate. Again, both lap and butt joints may have the rivets arranged in the form of chain riveting (Fig. 46*b*) or in the form of staggered riveting (Fig. 46*d* and 46*e*).

Definitions.—The terms defined below will be used frequently in the subsequent articles.

The *pitch* of a row of rivets is the distance between the centers of any two adjacent rivets in the row. The pitch is not necessarily the same for all rows. Thus, in Fig. 46*e* the pitch for the inner rows is denoted by p and for the other row by p_1 in which p_1 equals $2p$.

The *transverse pitch* is the distance between the center lines of two rows of rivets, denoted by p_t in Fig. 46(*d*), and the *diagonal pitch* in staggered riveting is the distance from the center of a rivet in one row to that of the nearest rivet in the next row, denoted by p_d in Fig. 46(*d*).

The *length of a repeating group* of rivets is used in connection with boiler and pipe joints to denote the shortest distance along the joint that includes a characteristic group of rivets which recurs along the length of the joint. The length of the repeating group is usually equal to the pitch, the maximum pitch being used when the pitches of all rows are not the same. In computations it is convenient to use the


p
p
p

The term *efficiency* of a riveted joint is used in connection with boiler and pipe joints to denote the ratio of the allowable strength of the joint to the allowable tensile strength of the solid plate.

15. Modes of Failure.—A riveted joint may fail in any of the following ways:

1. Shearing of the rivets as indicated in Fig. 47(*a*).

2. Rupturing of the plate on a section through a line of rivets or on a section along a diagonal pitch as indicated in Fig. 47(*b*).

3. Crushing of the rivets (or of the plate) due to the pressure of the plate on the side of the rivet (or of the rivet on the plate) as indicated in Fig. 47(*c*). This pressure is called the bearing pressure.

4. Shearing of the plate in the margin or tearing of the plate in the margin, as indicated in Fig. 47(*d*). A marginal failure is usually a combination of these two actions.

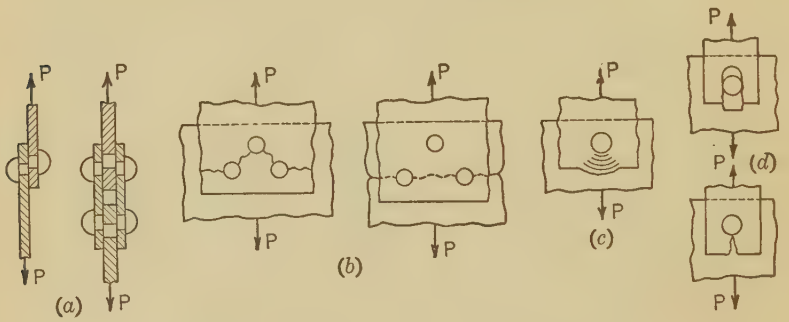


FIG. 47.—Modes of failure of riveted joints.

In joints having several rows of rivets the failure may be a combination of the above failures, as for example the rupturing of the plate along one row of rivets accompanied by the shearing of the rivets in another row.

The width of the margin can be fixed arbitrarily (except as it affects calking of a boiler joint to secure tightness) and experience has shown that if the width of the margin is $1\frac{1}{2}$ to 2 times the diameter of the rivet the joint will not fail in the margin. Further, experience has shown that failure along a diagonal will not occur if the transverse pitch is not less than $1\frac{3}{4}$ times the diameter of the rivet. It is evident therefore that if these conditions are satisfied the only stresses that need investigation in a riveted joint are (1) the shearing stress in the rivets, (2) the tensile stress in the plate on a section through a row of rivets, and (3) the bearing stress on the rivet (or on the plate).

16. Stresses in Riveted Joints.—As noted in Art. 13 the equation $P = as$ is used in the analysis of the stresses in a riveted joint in which

the load on the joint is assumed to act through the centroid of the rivet areas, the stress on each area then being assumed to be uniformly distributed.

In the equation $P = as$, as applied to riveted joints, P, for a structural joint, is the force that is transmitted from one member of a structure to another member, and, for a boiler or pipe joint it is usually the force transmitted through a repeating group of rivets.

The kind of stress s will be denoted by a subscript. Thus, s_s denotes the shearing unit-stress in the rivets; s_t the tensile unit-stress in the plate; and s_b the bearing unit-stress on the rivet. Similarly a_s denotes the shearing area on which s_s occurs; a_t the tensile area on which s_t occurs; and a_b the bearing area on which s_b occurs.

Other quantities used are defined as follows:

t is the thickness of the main plates or members;
d is the diameter of the rivets;
p is the pitch of the rivets.

To illustrate the method of determining the stresses in a riveted joint one particular joint will here be used; the method, however, is the same for all joints. Fig. 48(a) represents a repeating group of rivets in a double-riveted two-strap butt joint, in which it is desired to find the shearing, tensile and bearing unit-stresses in terms of the load P and the dimensions t, d, and p.

Shearing Unit-stress.—The load P is held in equilibrium, as shown in Fig. 48(b), by the total resisting shearing stress, which is the product of the shearing unit-stress s_s and the area a_s on which the shearing stress occurs. And since each of the two rivets has two shearing areas, a_s is equal to $\frac{4\pi d^2}{4}$. Hence

$$P = a_s s_s = 4\frac{\pi d^2}{4} s_s \quad \text{or} \quad s_s = \frac{P}{\pi d^2}. \quad . \quad . \quad . \quad . \quad (15)$$

It should be noted that if the above joint were a *lap* joint, each rivet would have only one shearing area, and hence the total area would be $2\frac{\pi d^2}{4}$ instead of $4\frac{\pi d^2}{4}$.

Tensile Unit-stress.—The total tensile resisting stress, $a_t s_t$, in the plate which holds the load P in equilibrium (see Fig. 48c) is the product of the area a_t, or $(p-d)t$, and the tensile unit-stress s_t. Hence

$$P = a_t s_t = (p - d)ts_t \quad \text{or} \quad s_t = \frac{P}{(p - d)t}. \quad . \quad . \quad . \quad (16)$$

Bearing Unit-stress.—The total bearing stress of the plate on the rivets which resists the load P (Fig. 48*d*) is the product of the bearing unit-stress s_b and the bearing area a_b on which the stress is assumed to be uniformly distributed. The area of contact between the rivet and plate is a semi-cylindrical area and the intensity of the pressure and the direction of the pressure probably vary greatly over the area, but the manner of variation is unknown. Only the component pressure parallel to the load P resists this load, and it is usually assumed that this component pressure is uniformly distributed over an area equal to the projec-

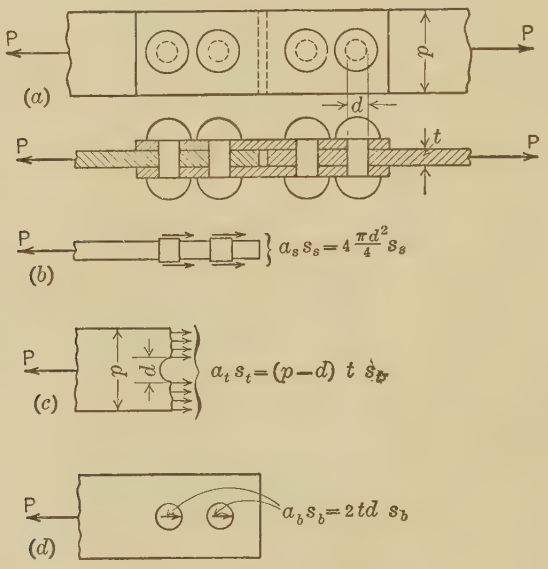


FIG. 48.—Stresses in riveted joint.

tion of the semi-cylindrical area of contact on a plane perpendicular to the direction of P, that is, an area equal to the product of the diameter of the rivet and the thickness of the plate. Hence, as indicated in Fig. 48*d*,

$$P = a_b s_b = 2tds_b \quad \text{or} \quad s_b = \frac{P}{2td}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

As noted above, the load P in boiler joints usually is taken as the load that is resisted by a repeating group of rivets, and in structural joints it is the total force transmitted from one member to another.

ILLUSTRATIVE PROBLEMS

Problem 68.—The members of the Fink truss shown in Fig. 49(*a*) are connected by riveted joints. The arrangement of the joint at *C* is shown in Fig. 49(*b*). The stress in members *BC* and *DC* are found to be 6930 lb. compression and 6930 lb. tension, respectively. The rivets are $\frac{3}{4}$ in. in diameter. Find the shearing unit-stress in the rivets connecting the members *BC* and *DC* to the gusset plate, and in the rivets connecting the gusset plate to the lower chord *ACEG*. Also find the bearing unit-stress of the members *BC* and *DC* on the rivets.

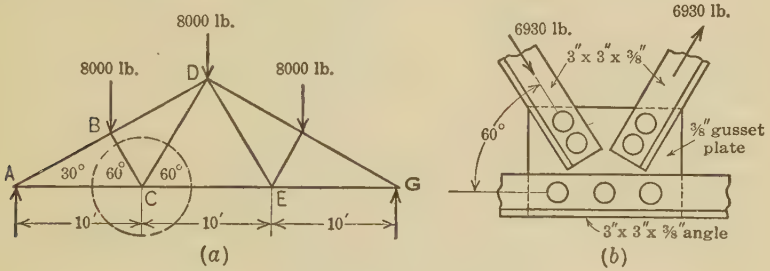


FIG. 49.—Riveted joint in truss.

Solution.—The shearing unit-stress in the rivets connecting members *BC* and *DC* to the gusset plate is

$$s_s = \frac{P}{a_s} = \frac{6930}{2 \times 0.442} = 7840 \text{ lb. per sq. in.},$$

the area of cross-section of a $\frac{3}{4}$-in. rivet being 0.442 sq. in.

The bearing unit-stress of the members *BC* and *DC* (or of the gusset plate) against the rivets is

$$s_b = \frac{P}{a_b} = \frac{6930}{2 \times \frac{3}{4} \times \frac{3}{8}} = 12{,}300 \text{ lb. per sq. in.}$$

The shearing unit-stress in the rivets that connect the gusset plate to the lower chord is

$$s_s = \frac{P}{a_s} = \frac{2 \times 6930 \cos 60°}{3 \times 0.442} = \frac{6930}{1.326} = 5230 \text{ lb. per sq. in.}$$

The bearing unit-stress of the lower chord on the three rivets is

$$s_b = \frac{P}{a_b} = \frac{6930}{3 \times \frac{3}{4} \times \frac{3}{8}} = 8210 \text{ lb. per sq. in.}$$

Problem 69.—A boiler having a diameter, *D*, of 72 in. is designed to resist an internal steam pressure, *R*, of 120 lb. per sq. in. The longitudinal joint is a double-riveted lap joint. The thickness, *t*, of the plates is $\frac{7}{16}$ in., the diameter, *d*, of the rivets is $\frac{7}{8}$ in., and the pitch, *p*, is $3\frac{1}{2}$ in. Find the shearing, tensile, and bearing unit-stresses in the joint. Also calculate the efficiency of the joint, assuming the allowable shearing, tensile, and bearing unit-stresses (in lb. per sq. in.) to have the following values: $s_s = 10{,}000$, $s_t = 15{,}000$, $s_b = 22{,}500$ (see Art. 18).

Solution.—According to Art. 12 and as indicated in Fig. 50(*a*) the force P transmitted through the joint for a pitch distance p of $3\frac{1}{2}$ in. along the joint, that is, through a repeating group of rivets (two rivets) is

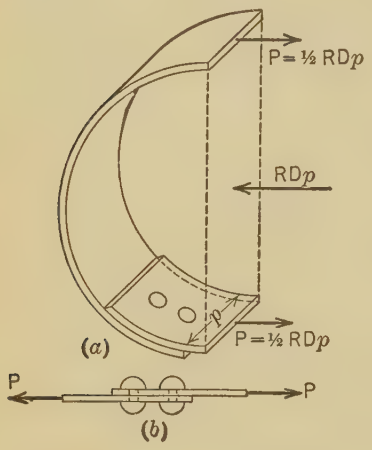


FIG. 50.—Stress in boiler joint.

$$P = \tfrac{1}{2}RDp$$
$$= \tfrac{1}{2} \times 120 \times 72 \times 3\tfrac{1}{2}$$
$$= 15{,}120 \text{ lb.}$$

The shearing unit-stress in the rivets, then, is

$$s_s = \frac{P}{2\frac{\pi d^2}{4}} = \frac{15{,}120}{2 \times 0.60} = 12{,}600 \text{ lb. per sq. in.}$$

The tensile unit-stress in the plate on a section through either row of rivets is

$$s_t = \frac{P}{(p - d)t} = \frac{15{,}120}{(3\frac{1}{2} - \frac{7}{8}) \times \frac{7}{16}} = 13{,}200 \text{ lb. per sq. in.}$$

The bearing unit stress of the plates against the rivets is

$$s_b = \frac{P}{2 \cdot d \cdot t} = \frac{15{,}120}{2 \times \frac{7}{8} \times \frac{7}{16}} = 19{,}800 \text{ lb. per sq. in.}$$

$$\text{Efficiency} = \frac{\text{least allowable strength of joint}}{\text{allowable strength of solid plate}}.$$

$$\text{Shearing efficiency } e_s = \frac{2\frac{\pi d^2}{4} \times 10{,}000}{p \cdot t \cdot 15{,}000} = \frac{\pi d^2}{2p \cdot t} \times \frac{2}{3} = 0.524 = 52.4 \text{ per cent.}$$

$$\text{Tensile efficiency } e_t = \frac{(p - d)t\,15{,}000}{p \cdot t \cdot 15{,}000} = \frac{p - d}{p} = 0.75 = 75 \text{ per cent.}$$

$$\text{Bearing efficiency } e_b = \frac{2 \cdot dt \cdot 22{,}500}{p \cdot t \cdot 15{,}000} = \frac{2d}{p} \times \frac{3}{2} = 0.75 = 75 \text{ per cent.}$$

Therefore, the efficiency of the joint is 52.4 per cent. If the joint were well proportioned the three efficiencies would be more nearly equal.

PROBLEMS

70. A boiler 80 in. in diameter is designed to withstand a pressure of 50 lb. per sq. in. The longitudinal joint is a single-riveted lap joint; the rivets have a pitch of 3 in.; the plates are $\frac{3}{8}$ in. thick and the rivets are $\frac{7}{8}$ in. in diameter. Find the shearing, tensile, and bearing unit-stresses in the joint. Also find the efficiency of the joint, using the allowable unit-stresses given by the Boiler Code in Art. 18.

71. A column bracket (Fig. 51) consisting of a 6-in. by 6-in. by $\frac{3}{8}$-in. angle carries a load of 20,000 lb. and is riveted with five rivets $\frac{7}{8}$ in. in diameter to a 12-in., 30-lb. channel which forms part of a column. The thickness of the web of the channel is $\frac{1}{2}$ in. Find the shearing unit-stress in the rivets and the bearing unit-stress of the angle on the rivets.

Ans. s_s = 6670 lb. per sq. in., s_b = 12,200 lb. per sq. in.

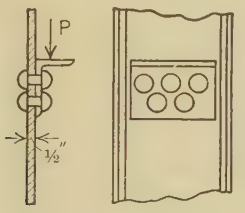


FIG. 51.—Column bracket.

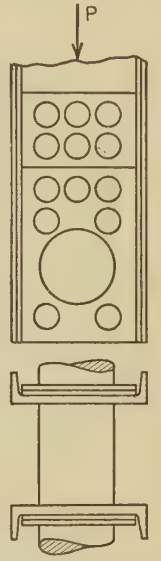


FIG. 52. — Column bearing on pin.

72. A bridge post, Fig. 52, consists of two 10-in., 25-lb. channels latticed together. The total compressive stress P in the post is 275,000 lb. The post transmits this stress to a pin 6 in. in diameter by means of the bearing on the two channel webs and on two $\frac{5}{16}$-in. pin-plates riveted to each channel, the thickness of each web being 0.53 in. The rivets are $\frac{7}{8}$ in. in diameter. Find the shearing unit-stress in the rivets (*a*) on a section between the outer plate and the inner plate, (*b*) between the web of the channel and the plate next to it. Assume that the pressure per unit-area of the pin on the plates is the same as that on the web. Hint: the pressure of the pin on each outer plate is resisted by the shearing stress in seven rivets, and the pressure of the pin on each pair of plates is resisted by the shearing stress in thirteen rivets.

Ans. (*a*) 8840 lb./in.^{2}; (*b*) 9520 lb./in.^{2}

73. The plates of a tank 60 in. in diameter are $\frac{3}{4}$ in. thick and are spliced by means of a double-riveted butt joint with two strap-plates. The strap-plates are $\frac{3}{8}$ in. thick. The rivets are staggered; the two lines are 3 in. apart, and the pitch on each line is $3\frac{1}{2}$ in. The rivets are $\frac{7}{8}$ in. in diameter. If the allowable unit-stresses are: s_s = 10,000 lb. per sq. in., s_t = 15,000 lb. per sq. in., and s_b = 22,500 lb. per sq. in., what is the maximum internal pressure to which the tank can be subjected. *Ans.* R = 228 lb. per sq. in.

74. A boiler 60 in. in diameter resists an internal pressure of 180 lb. per sq. in. The plates are $\frac{3}{4}$ in. thick and are spliced by means of a double-riveted butt joint with two strap-plates. The strap-plates are $\frac{3}{8}$ in. thick and the rivets are staggered; the two lines are 3 in. apart and the pitch on each line is also 3 in. The rivets are $\frac{7}{8}$ in. in diameter. Find the shearing, bearing and tensile unit-stresses.

Ans. s_s = 6740 lb. per sq. in., s_b = 12,350 lb. per sq. in., s_t = 10,160 lb. per sq. in.

75. In Fig. 53(*b*) is shown a joint that occurs in a riveted Pratt truss as at D in Fig. 53(*a*). If the rivets are $\frac{3}{4}$ in. in diameter and the allowable shearing unit-stress is 10,000 lb. per sq. in., what maximum value can P_1 have? When P_1 has this

maximum value what is the value of P_2 if the height of the truss is 30 ft. and each panel length is 20 ft.? What is the bearing unit-stress of the gusset plate on the rivets in each member if the gusset plate is $\frac{3}{8}$ in thick?

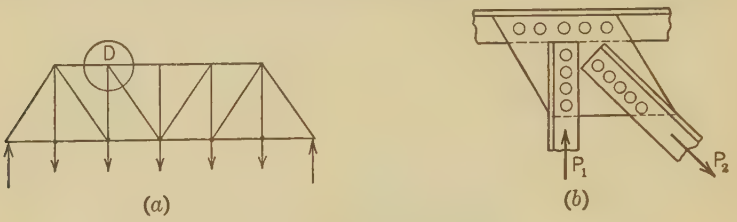


FIG. 53.—Riveted joint in truss.

76. A triple-riveted lap joint is made of $\frac{9}{16}$ in. plates. The diameter of the rivets is 1 in.; the pitch in the outer rows is $4\frac{1}{2}$ in.; the pitch in the inner row is $2\frac{1}{4}$ in. and the distance between the rows of the rivets is $2\frac{1}{2}$ in. (*a*) What is the efficiency of this joint? (*b*) If this joint is the longitudinal seam of a 90-in. tank, what is the maximum allowable internal pressure? Use values for allowable unit-stresses stated in Prob. 69 and Art. 18. Hint: Investigate for failure by tension in outer row alone, and on the inner row alone as well as for failure by shearing of all the rivets.

Ans. $e_t = 74.1\%$, $R = 139$ lb. per sq. in.

77. A boiler 100 in. in diameter, having plates $\frac{3}{4}$ in. thick, is subjected to an internal steam pressure of 160 lb. per sq. in. The longitudinal joint is a double-riveted butt joint with two cover plates. The diameter of the rivets is $\frac{7}{8}$ in. and the pitch is 3 in. Find the bearing, shearing, and tensile unit-stresses in the joint.

78. A tensile member carrying an axial load of 48,000 lb. consists of two rectangular plates 8 in. wide by $\frac{5}{8}$ in. thick connected by a lap joint. (*a*) Find the minimum number of rivets required if the diameter of each rivet is $\frac{7}{8}$ in. (*b*) What is the greatest number of rivets that may be placed in one transverse row? Use working unit-stresses as stated in Prob. 69. *Ans.* (*a*) 8; (*b*) 3.

17. Assumptions. Conditions Affecting Strength of Riveted Joints.—*Friction.*—Since the rivets in a joint shrink while cooling, the plates are drawn tightly together which in turn causes friction between the plates when the joint is stressed. The amount of the friction however is indeterminate, and tests [1] have shown, that slipping may occur at ordinary working stresses. Hence the assistance obtained from the friction is usually neglected in the design of joints.

Bending and Tension.—In the design of a riveted joint, the assumption is usually made that only shearing stresses exist on a cross-section of a rivet. The rivet, however, is always subjected to more or less bending, and the bending may have an important influence on the stresses in long rivets that connect several plates as indicated in Fig. 54. And, in a simple lap joint, bending also occurs as is indi-

[1] Bulletin 49, Engineering Experiment Station, University of Illinois.

cated in Fig. 55. The effect of bending, however, is not disastrous if the material is ductile. Its influence in the design of the simpler types of joints is taken account of to some extent in selecting the values of the working unit-stresses.

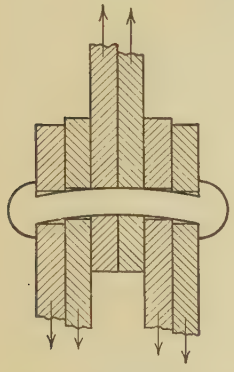


FIG. 54.—Bending in rivet.

Occasionally rivets must resist a direct tensile load as in the arrangement shown in Fig. 56. The allowable tensile stress in rivets should always be low and the thickness of the head of the rivet should be investigated to see if the shearing area in the head is ample; the contraction of the rivet during cooling causes stress in the rivet of unknown amount,

FIG. 55.—Bending in rivet.

but sufficient sometimes to cause the rivet head to snap off. Hence, rivets are not considered desirable for resisting direct tension.

Rivets are cheaper than bolts, otherwise bolts would be more generally used since they can be made safe in tension as well as in shear. However, with the vibration that occurs in many structures it is difficult to keep the nuts of the bolts tight, and even the best nut-locks should be inspected frequently.

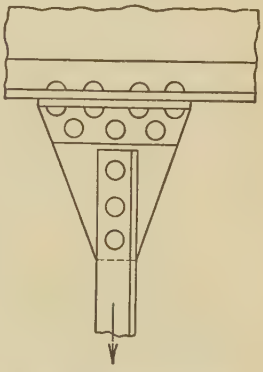


FIG. 56.—Rivets subjected to tensile stress.

Rivet Holes.—Rivet holes are usually formed by punching the plates cold, for, although holes made by drilling are preferable, punching is the cheaper process. There are two objections to punching the hole: (1) The holes seldom come into exact alignment or register, and in order to cause the holes to register it is frequently necessary to ream out one or both holes or to use a drift pin. The latter method is particularly objectionable, but by either method the holes are enlarged and the rivets may not fill the holes, thereby causing excessive stress on some of the other rivets; (2) the tensile strength of the material around the hole is reduced due to the injury of the material accompanying the lateral flow of the metal caused by the punch.

Both of these difficulties may greatly be reduced by making the punched holes somewhat under-size and then reaming them to the correct size, after the plates or members are put together. Plates are seldom punched and used in their natural punched condition except in the case of thin plates and in the cheaper classes of work. The injury done to the plate by punching may be partially remedied by thorough annealing; it is generally impracticable, however, to anneal portions of large sheets merely for the sake of the rivet holes. Splice bars used in connecting railroad rails are frequently punched cold and then the whole bar annealed instead of being punched hot. In the best classes of marine work it is specified that all rivet holes shall be drilled from the solid plate, but the specifications for most structural work and for many pressure vessels require the holes to be punched small and reamed to the correct size. In obtaining the tensile unit-stress in a plate in which the holes are punched to size (not reamed) the diameter of the hole is sometimes assumed to be $\frac{1}{8}$ inch larger than that of the punched hole.

Methods of Riveting.—In structural work the riveting of some of the joints is done in the shop, whereas other joints must be riveted in the field during the erection of the structure. Shop riveting is usually done by machines which press the rivet in place and when well done is better than hand riveting. Since the rivet is a little smaller than the hole, the rivet while being driven must be expanded throughout its whole length if it is to fill the hole, which is a necessary requirement for a good joint. Large rivets, therefore, almost always are machine riveted in order to obtain the heavy pressures required.

Specifications usually require that the allowable unit-stresses for hand-driven field rivets shall be less (often one-third less) than for machine-driven shop rivets since the conditions in the field are difficult to control. For example, the heating of the rivets varies; some rivets are burned, others are underheated and may be too cool when driven to be made to fill the hole, etc.

In designing joints the rivet is assumed to fill the hole and hence the stress, in compression members, is not increased due to having rivet holes in the member as is the case in tension members.

Distribution of Load among Rivets.—If the load on a joint is a central load (passes through the centroid of the group of rivets) the assumption is made that the load is distributed equally to all the rivets (see Art. 65 for a discussion of eccentric loading). Obviously, it is impossible for the load to be distributed in this way, since the load on the rivets in one row depends on the yielding of the rivets in the row nearer to the load and also on the yielding of the plate between the rows, etc. The distribution of the load, therefore, is not easily determined, particularly

in joints having several rows of rivets. But, in the absence of definite information the assumption stated above is used, particularly in the case of the simpler types of joints.

18. Allowable Stress.—The discussion, in the preceding articles, of the uncertainties in the actions occurring in joints should make it evident that the selection of the working or allowable unit-stresses in shear, tension, and bearing should be based not only on the strength of the materials of which the joint is made, but also on the results of tests of actual joints. Many tests on riveted joints have been made, and tests and experience indicate that in the design of joints of the usual proportions and with the methods of calculating stresses already discussed, the values of working stresses given below may be used as average values. In using these values it is understood that the members or plates to be riveted meet the specifications for structural steel or boiler plate, etc., and that the rivets meet a similar requirement for rivet steel. Further, it is assumed that the rivet holes are punched small and reamed to size or are drilled from the solid plate, and that the rivets are shop driven. For joints having field-driven rivets or having holes punched to size (not reamed) the values given below should be reduced.

Shearing unit-stress in rivets;	s_s = 10,000 lb. per sq. in.
Tensile unit-stress in plate;	s_t = 15,000 lb. per sq. in.
Bearing unit-stress on rivets;	s_b = 22,500 lb. per sq. in.
Bearing unit-stress on pins;	s_b = 22,500 lb. per sq. in.
Compressive unit-stress in plate;	s_c = 15,000 lb. per sq. in.

Thus the ratios of the shearing and bearing allowable unit-stresses to the tensile allowable unit-stress are as follows:

$$\frac{s_s}{s_t} = \frac{2}{3} \quad \text{and} \quad \frac{s_b}{s_t} = \frac{3}{2}.$$

These ratios are convenient for use in calculations in design.

Other Values of Working Stresses.—The following values of working stresses (in lb. per sq. in.) are recommended, for riveted joints in highway bridges, by a special committee of the American Society of Civil Engineers:

s_s in pins and shop-driven rivets	= 12,000
s_s in power-driven field rivets	= 10,000
s_b on pins and shop-driven rivets	= 24,000
s_b on power-driven field rivets	= 20,000
s_t on net section	= 16,000

The Boiler Code of the American Society of Mechanical Engineers

gives the following values of working stresses (in lb./in.^{2}) for riveted joints in boilers:

s_s in rivets = 8,800
s_b on rivets = 19,000
s_t in plate = 11,000

PROBLEMS

79. In an ideal boiler or pipe joint the allowable strength (not allowable unit-stresses) in shear, bearing, and tension are equal. (*a*) By equating the allowable shearing strength to the allowable bearing strength show that the diameter of the rivets in a *lap* joint is 2.87 times the thickness of the plate ($d = 2.87t$), if the first set of values for working stresses given in the above article is used. (*b*) Similarly, show that in an ideal butt joint having two cover plates, $d = 1.43t$. Which of these two types of joints is better suited for boilers, pipes, etc., having thick plates? Hint: If n represents the number of rivets we have for (*a*)

$$n \cdot \frac{\pi d^2}{4} s_b = ntds_b \quad \text{or} \quad d = \frac{4}{\pi} \frac{s_b}{s_s} \cdot t.$$

80. By equating the allowable tensile strength to the allowable bearing strength show that the pitch in an ideal double-riveted lap joint is four times the diameter of the rivets ($p = 4d$), using the first set of values of working stresses given in the above article. Also show that the efficiency of an ideal double-riveted lap joint is 75 per cent.

81. Find the relation between p and d, and also the efficiency, of an ideal double-riveted butt joint having two cover plates.

82. What would be the relation between d and t in Prob. 79 if the values for working stresses given by the Boiler Code were used?

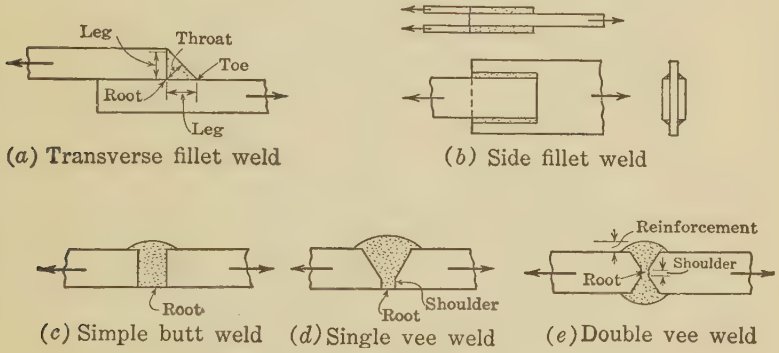
19. Welded Joints.—Welding is defined as a localized consolidation of metals by means of heat. The greater part of structural welding is done by the fusion method using either an oxy-acetylene torch or an electric arc. The welding material in the form of welding rod or wire melts and unites with the base metal to form the weld.

The two main classes of welds are fillet welds and butt welds. These two types of welds with the names of the various parts of the welds are shown in Fig. 57. As yet no simple method for determining the reliability of a welded joint has been devised. X-ray and Gamma ray examination of welded joints are used on important work, such as in pressure vessels, for determining defects in the material; and welders are required to pass tests in order to qualify for the work of welding.

Allowable Stresses for Welded Joints.—The code for fusion welding of the American Welding Society gives the following permissible unit stresses:

Shear on section through throat of weld.....11,300 lb. per sq. in.
Tension on section through throat of weld.....13,000 lb. per sq. in.
Compression on section through throat of weld.....15,000 lb. per sq. in.

Fillet welds placed transverse to the direction of the load (Fig. 57*a*) shall be considered as under shear as well as the side fillet weld (Fig. 57*b*), the minimum section in each weld being the section through the throat. Thus for a fillet weld with each leg $\frac{3}{8}$ in. long, the throat section per inch of length of joint is $0.707 \times \frac{3}{8} = 0.266$ sq. in., and the allowable shearing load per inch of joint is $11{,}300 \times 0.266 = 3000$ lb. For various sized fillets we have:



(*a*) Transverse fillet weld (*b*) Side fillet weld

(*c*) Simple butt weld (*d*) Single vee weld (*e*) Double vee weld

FIG. 57.—Types of welds.

Leg of Fillet, Inches	Allowable Shearing Stress per Lineal Inch of Joint Pounds
$\frac{1}{2}$	4000
$\frac{7}{16}$	3500
$\frac{3}{8}$	3000
$\frac{5}{16}$	2500
$\frac{1}{4}$	2000

The simple butt weld (Fig. 57*c*) is not usually employed on material over $\frac{1}{4}$ in. thick; the single Vee weld is used mainly on material $\frac{1}{4}$ to $\frac{1}{2}$ in. thick, and the double Vee weld on material over $\frac{1}{2}$ in. thick.

ILLUSTRATIVE PROBLEM

Problem 83.—Let it be required to weld a $3'' \times 3'' \times \frac{3}{8}''$ rolled steel angle to a gusset plate using fillet welds as indicated in Fig. 58(*a*), so that the full tensile strength of the angle is developed, assuming a tensile working stress of 18,000 lb. per sq. in. in the angle.

Solution.

Area of angle section = 2.11 sq. in.

Allowable tensile load on angle = $2.11 \times 18{,}000 = 38{,}000$ lb.

Use a $\frac{5}{16}$-in. fillet for which, as shown above, the allowable stress per lineal inch = 2500 lb.

$$\text{Total length of fillet} = \frac{38{,}000}{2500} = 15.2 \text{ in.}$$

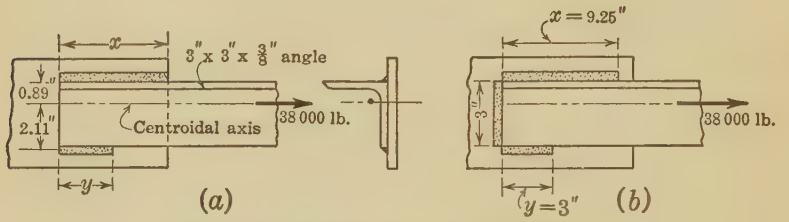


FIG. 58.—Members joined by fillet welds.

It is customary to assume that the length of the weld on the two sides of the angle bar are such that the total (resultant) resisting stress acts along the centroidal axis of the angle bar, and hence that the lengths of the welds on the two sides are inversely proportional to the distances from the centroidal axis. Thus:

$$\frac{x}{y} = \frac{2.11}{0.89} \quad \text{or} \quad x = 2.37y; \quad x + y = 3.37y = 15.2 \text{ in.}$$

Hence

$$y = \frac{15.2}{3.37} = 4.5 \text{ in., length of weld along the toe}$$

$$x = 2.37y = 10.7 = 10\tfrac{3}{4} \text{ in. (say)}$$

If this gives too long a gusset plate, the plate may be shortened by welding along the back end of the angle giving the design shown in Fig. 58(*b*), in which the total resisting stress of the weld is made to act along the centroidal axis of the angle bar.

PROBLEMS

84. Design a fillet-welded joint for attaching two $3\frac{1}{2}$ in. $\times$ $2\frac{1}{2}$ in. $\times$ $\frac{5}{16}$ in. angles to a gusset plate as indicated in Fig. 59, assuming an allowable tensile stress in each angle of 18,000 lb. per sq. in. Use a weld with each leg $\frac{5}{16}$ in. long. Area of each angle section is 1.78 in.^{2} *Ans.* 4.17 in. at toe; 8.63 in. at heel.

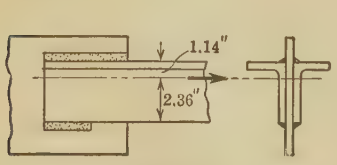


FIG. 59.

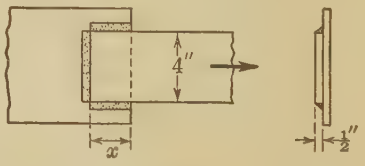


FIG. 60.

85. A steel plate $\frac{1}{2}$ in. thick and 4 in. wide is attached to another plate by means of a fillet weld as indicated in Fig. 60. What is the minimum allowable length x of

weld assuming the allowable unit-stress in the plate to be 16,000 lb. per sq. in.? Each leg of the fillet weld is 0.5 in.

86. Two plates 6 in. $\times \frac{5}{8}$ in. are to be joined by means of a butt weld. What type of butt joint would probably be used, and what should be the thickness of the weld at its center if the allowable tensile unit-stress in the plate is 18,000 lb. per sq. in.?

87. Determine the minimum length of the weld on each side of the angles that would be required for a fillet-welded joint to replace the riveted joint shown in Fig. 49(*b*). Assume that each leg of the weld is $\frac{5}{16}$ in. *Ans.* 0.822 in., 1.95 in.

REVIEW QUESTIONS AND PROBLEMS

88. Draw a free body diagram of one-half of a boiler shell and by use of this diagram explain the meaning of the equation $RD = 2t \cdot s_t$.

89. Show that the tensile unit-stress in a boiler shell on a longitudinal section is twice that on a transverse section.

90. Draw free body diagrams showing the external forces and the internal shearing, tensile, and bearing forces acting on a double-riveted lap joint (similar to those in Fig. 48 for a butt joint), and by use of these diagrams write the expressions for the shearing, tensile, and bearing unit-stresses in terms of the load P transmitted through the joint, and the dimensions d, t, and p of the joint.

91. A high-strength steel ring or hoop having a proportional limit of 100,000 lb. per sq. in. and having an inside diameter of 39.9 in. and a thickness of $\frac{1}{4}$ in. is heated until it just fits over a smooth rigid cylinder which is at room temperature. The diameter of the cylinder is 40 in. After the ring has cooled to room temperature (*a*) what is the stretch of the ring per unit of length, assuming that the diameter of the cylinder does not change; (*b*) what is the tensile unit-stress in the ring; and (*c*) what is the pressure per unit area of the cylinder on the ring?

Ans. (*a*) 0.0025; (*b*) 75,000 lb. per sq. in.; (*c*) 937 lb. per sq. in.

92. A steam boiler 60 in. in diameter has a longitudinal double riveted butt joint with two cover plates. The rivets are $\frac{3}{4}$ in. in diameter and the pitch is $2\frac{1}{2}$ in. The thickness of the boiler plate is $\frac{1}{2}$ in. (*a*) Calculate the allowable internal steam pressure using the working stresses given by the A.S.M.E. Boiler Code (see Art. 18). (*b*) Is the joint well designed from the viewpoint of strength?

Ans. (*a*) 128 lb. per sq. in.; (*b*) No; too strong in shear and too weak in tension, $e_s = 113\%$; $e_t = 70\%$.

93. Two rectangular plates each having a rectangular cross-section $\frac{3}{4}$ in. thick by 10 in. wide are joined by a riveted butt joint having two cover plates. A load of 84,000 lb. is transmitted from one plate to the other plate. It is specified that the rivets shall be $\frac{7}{8}$ in. in diameter and that the working unit-stresses shall be: $s_s = 10{,}000$ lb. per sq. in., $s_t = 16{,}000$ lb. per sq. in., and $s_b = 20{,}000$ lb. per sq. in. (*a*) How many rivets on each side of the joint are needed? (*b*) What is the greatest number of rivets that can be placed in one row parallel to the joint? *Ans.* (*a*) 7; (*b*) 3.

94. If the length of each of the four strips of weld in the welded joint shown in Fig. 57(*b*) is 4 in. and each leg of each weld is $\frac{3}{8}$ in., what is the least thickness the two steel side plates can have if their widths are 3 in. and if the allowable tensile stress in each plate is 16,000 lb. per sq. in.? *Ans.* $\frac{1}{2}$ in.

CHAPTER III

STRESS AND STRAIN CAUSED BY TORSIONAL LOADS

20. Torsional Load, Twisting Moment, and Resisting Moment Defined.—*Torsional Load.*—Forces that cause a bar to twist about its central axis are called *torsional loads.* The resultant of torsional loads acting on a shaft is a twisting couple (two equal, opposite, non-collinear forces) since simple turning or twisting can be produced only by a couple. The resultant moment (or moment of the resultant couple) which causes the twisting is the algebraic sum of the moments of the torsional loads about the axis of the bar or shaft.

Twisting Moment.—The *twisting moment* for a section of a shaft is the name given to the algebraic sum of the moments, about the axis of the shaft, of the torsional loads that lie to one side of the section. The symbol T will be used to denote the twisting moment.

It is here assumed that the torsional loads lie in planes perpendicular to the axis of the shaft. If the forces do not lie in such planes the forces may be resolved into components perpendicular and parallel, respectively, to the axis, the components parallel to the axis being neglected in the following discussion.

Resisting Moment.—The resisting moment at any section of a bar or shaft is the algebraic sum of the moments about the axis of the shaft, of the stresses that act on the section; the resisting moment holds in equilibrium the twisting moment.

21. The Torsion Formula.—Let it be required to find the relation between the twisting moment acting on a cylindrical shaft, the shearing unit-stress at any point in the shaft on a section perpendicular to the axis of the shaft, and the dimensions of the area of the section. In Fig. 61(*b*) is shown a shaft subjected to a twisting moment T (equal to Pp) at one end of the shaft. Now since the shaft is in equilibrium the body A must exert an equal and opposite moment (not shown) on the other end of the shaft and these two moments cause the shaft to twist. Now let a plane BC be passed through the shaft and let the portion of the shaft to the right of the plane be removed. If the remaining (left) portion of the shaft (Fig. 61*a*) is held in equilibrium (as it was when a part of the whole shaft) a moment equal and opposite to T must be applied

to it. This resisting moment (denoted by T_r, Fig. 61*a*) is exerted on the left portion by the right portion on the section DE cut by the plane, and is due to the shearing stresses developed on the section. Thus from the condition of equilibrium, we have,

Moment of external forces = Moment of internal forces or stresses, that is,

Twisting moment = Resisting moment, or $T = T_r$.

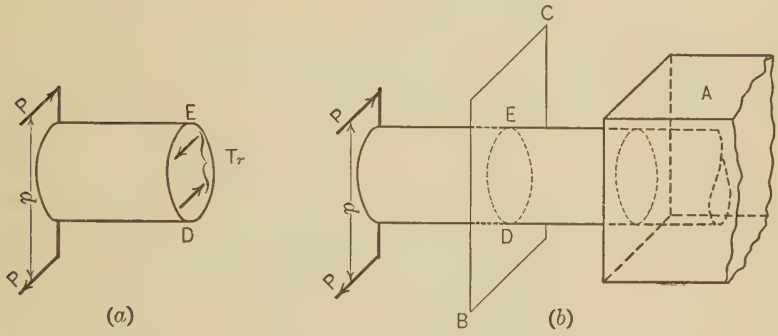


FIG. 61.—Cylindrical bar subjected to a twisting moment.

Before the resisting moment T_r can be expressed in terms of the unit-stress at any point in the area and the dimensions of the area, the manner in which the unit-stress varies over the area must be known. It is assumed that the shearing unit-stress is zero at the center of the section and that it increases directly as the distance from the center as indicated in Fig. 62 (the justification for this assumption is discussed in the next article). This assumption may be expressed mathematically as follows:

$$\frac{s_{s\rho}}{s_{sc}} = \frac{\rho}{c} \quad \text{or} \quad \frac{s_{s\rho}}{\rho} = \frac{s_{sc}}{c} = \text{etc.} = \text{a constant}, \quad . \quad . \quad . \quad (18)$$

in which $s_{s\rho}$ and s_{sc} are the shearing unit-stresses at points whose distances from the center of the section are ρ and c, respectively. The resisting moment may now be found as follows (see Fig. 62):

Total shearing stress on an element of area da at the distance ρ from center of shaft = $s_{s\rho}da$.

Moment of this total stress about center of shaft = $s_{s\rho}da \cdot \rho$.

Sum of moments of stresses on all elements of area = resisting moment = $T_r = \int s_{s\rho}da\rho$.

This may be written

$$T_r = \int \frac{s_{s\rho}}{\rho} \cdot \rho^2 da.$$

But

$$T = T_r \text{ and } \frac{s_{s\rho}}{\rho} = \frac{s_{sc}}{c} = \text{a constant.}$$

Therefore,

$$T = \frac{s_{s\rho}}{\rho} \int \rho^2 da,$$

in which $\int \rho^2 da$ is the polar moment of inertia of the area with respect to the central axis of the shaft. It will be denoted by the symbol J; it is equal to $\frac{\pi d^4}{32}$ (see Art. 171 and Prob. 498, App. I).

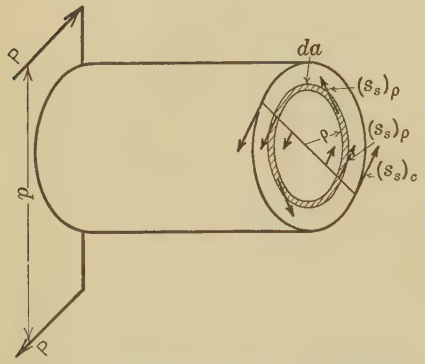


FIG. 62.—Shearing stresses in cylindrical shaft.

If now for convenience we let s_s, instead of $s_{s\rho}$, denote the shearing unit-stress on the area da at the distance ρ from the center, where ρ may be any value from 0 to the radius of the shaft, we may write

$$T = \frac{s_s J}{\rho} \text{ or } s_s = \frac{T\rho}{J}. \quad . \quad . \quad (19)$$

which is called the *torsion formula*. But since the maximum value of s_s is usually involved in the use of the formula, the value of ρ will usually be the radius of the shaft, and hence the formula is usually written

$$T = \frac{s_s J}{c}.$$

If s_s is expressed in pounds per square inch, T must be expressed in pound-inches, c in inches and J in inches to the fourth power (in.^{4}).

22. Justification for Assumption of Stress Distribution.—In order to supply the evidence and reasoning leading to the assumption that the shearing unit-stress at any point in the section of the cylindrical shaft varies directly as the distance of the point from the center of the section, it is necessary to consider: (*a*) the manner in which the shearing unit-strains of the fibers of the shaft vary when the shaft is twisted,

and (*b*) the relation between the shearing unit-strain at a point in a material and the shearing unit-stress at the point.

Fig. 63(*a*) represents a cylindrical shaft twisted by the twisting moment T or P_p.

It is assumed that when a cylindrical [1] shaft is twisted, a normal plane section of the shaft before twisting is approximately a plane section after twisting, and it is assumed also that a diameter in the section before twisting is a diameter (straight line) after twisting.[2] An outer element or fiber AB, having a cross-sectional area da, takes the form of a helix AB' when the shaft is twisted. Now a small portion of the fiber, which is at C before the shaft is twisted is at C' after the shaft has been

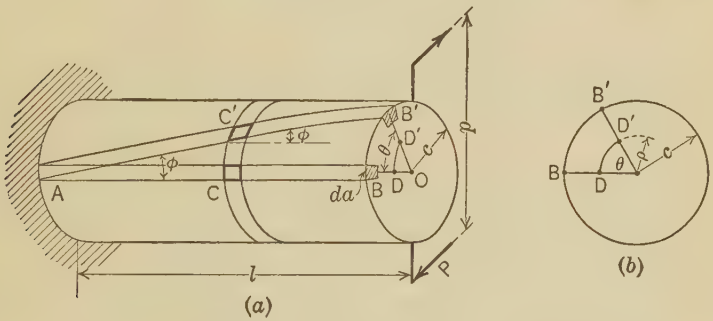


FIG. 63.—Shearing strain in shaft due to twisting moment.

twisted and is deformed as shown in the figure by the shearing stresses on its faces.

Further, as discussed in Art. 5, the shearing unit-strain, ϵ_s, at any point of a body is equal to the tangent of the angle of deviation from a right angle of two planes passing through the point. Therefore, the shearing unit-strain, ϵ_s, at the surface of a cylindrical shaft when twisted is

$$\epsilon_s = \tan \phi.$$

But $\tan \phi$ (Fig. 63*a*) is equal to the arc $\overline{BB'}$ divided by the length, l, of the shaft, and further, $\overline{BB'}$ is equal to the radius c times the angle of

[1] A plane section of a shaft whose cross-section is not circular does not remain plane when the shaft is twisted. Torsion of shafts having non-circular sections is discussed briefly in Chap. XIX.

[2] Although these assumptions seem reasonable, it is very difficult to obtain direct experimental verification. The justification for the assumptions is to be found in the agreement of results calculated from the formula based on these assumptions and experimental results.

twist, θ, expressed in radians. Therefore,

$$\epsilon_s, \text{ at surface, } = \tan \phi = \frac{BB'}{l} = \frac{c\theta}{l}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Similarly, since a diameter remains a straight line, the shearing unit-strain at a point at the distance ρ from the center of the shaft (Fig. 63*b*) is

$$\epsilon_s, \text{ at distance } \rho, = \frac{DD'}{l} = \frac{\rho\theta}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

By dividing eq. 20 by eq. 21 we have

$$\frac{\epsilon_s \text{ at distance } c \text{ from center}}{\epsilon_s \text{ at distance } \rho \text{ from center}} = \frac{c}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

That is, when a shaft is twisted the shearing unit-strains of the fibers vary directly as the distances of the fibers from the central axis of the shaft.

But as stated in Art. 6, the unit-stress at any point in the material is proportional to the corresponding unit-strain at that point provided that the unit-stress does not exceed the proportional limit of the material. Therefore, since the shearing unit-strains of the fibers are proportional to the distances of the fibers from the center of the shaft, and since the shearing unit-strains of the fibers are also proportional to the shearing unit-stresses on the fibers it follows that:

> The shearing unit-stress on a fiber of the shaft varies directly as the distance of the fiber from the axis of the shaft.

It is important to note that the torsion formula is applicable only to a right circular cylinder, solid or hollow, in which the shearing unit-stress does not exceed the shearing proportional limit, for otherwise the shearing unit-stresses on the fibers at any section will not vary as the distances of the fibers from the center of the shaft.

ILLUSTRATIVE PROBLEM

Problem 95.—A shaft (Fig. 64) having a diameter, d, of 1.5 in. rotates at constant speed. It has keyed to it a driving pulley A and two driven pulleys B and C. The effective pull for each belt (difference between tensions on the two sides of a pulley) is 40 lb. per in. of width of belt. The belt on pulley A is 6 in. wide, and on B 4 in. wide. Pulley A has a diameter of 36 in., B of 12 in., and C of 18 in. Find the maximum shearing unit-stress in the shaft between A and B; also between B and C.

Solution.—The twisting moment for any section between A and B is

$$T_1 = 40 \times 6 \times 18 = 4320 \text{ lb.-in.}$$

The maximum shearing unit-stress, s_s', in any section between A and B is

$$s_s' = \frac{T_1 c}{J} = \frac{T_1 \frac{d}{2}}{\frac{\pi d^4}{32}} = T_1 \frac{16}{\pi d^3}$$

$$= 4320 \times \frac{16}{\pi(1.5)^3} = 6520 \text{ lb. per sq. in.}$$

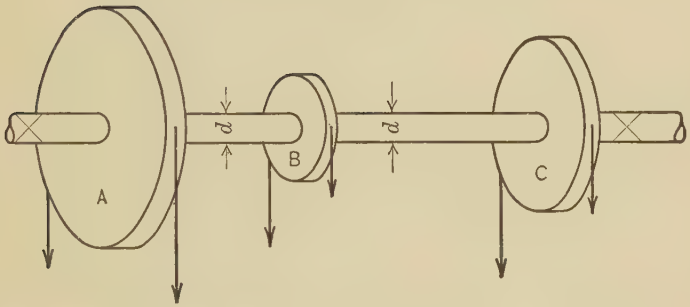


FIG. 64.—Shaft transmits twisting moment from pulley A to pulleys B and C.

Similarly, the maximum shearing unit-stress, s_s'', at any section between B and C is

$$s_s'' = \frac{T_2 c}{J} = (T_1 - 4 \times 40 \times 6) \times \frac{c}{J}$$

$$= 3360 \times \frac{16}{\pi d^3} = 5070 \text{ lb. per sq. in.}$$

PROBLEMS

96. If a twisting moment of 10,000 lb.-ft. is applied to a cylindrical shaft 4 in. in diameter, what shearing unit-stress will be developed in the outer fibers of the shaft? *Ans.* $s_s = 9550$ lb. per sq. in.

97. A bronze shaft 2 in. in diameter is twisted so that the shearing unit-stress in the fibers $\frac{1}{2}$ in. from the center of the shaft is 5000 lb./in.^{2} Calculate the twisting moment acting on the shaft.

98. The shearing yield-point of a certain steel shaft is 30,000 lb. per sq. in. and a working stress equal to one-third of the yield-point is specified. What should be the diameter of the shaft if the shaft is subjected to a twisting moment of 1200 lb.-in.? *Ans.* $d = 0.85$ in.

99. A workman sometimes breaks a bolt in tightening the nut. If the workman uses a wrench with a handle which gives him a moment arm of 15 in. and he exerts a force of 75 lb. on the handle, what should be the minimum diameter of a structural steel bolt to prevent the shearing proportional limit of the bolt from being exceeded? Neglect friction.

100. In the band brake shown in Fig. 65 the force P is 100 lb., the angle of contact, α, is 270°, and the coefficient of friction μ, is 0.2. Therefore, the tension in the band at C is $T_c = 1300$ lb. and, if the maximum friction is being developed, the tension in the band at B is $T_B = T_C \times e^{\mu\alpha} = 3330$ lb. If the diameter of the shaft is 3 in. what is the maximum torsional shearing unit-stress in the shaft?

Ans. $s_s = 3830$ lb. per sq. in.

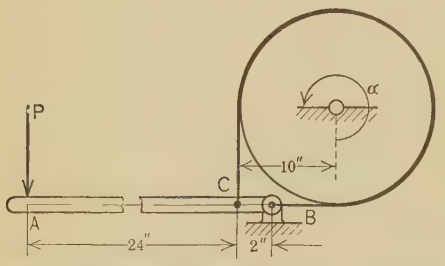


FIG. 65.—Band brake transmits twisting moment to shaft.

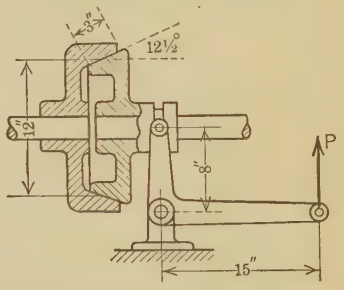


FIG. 66.—Friction clutch transmits twisting moment from one shaft to another.

101. A force P of 196 lb. (Fig. 66) will cause a normal pressure of 15 lb. per sq. in. on the rubbing surfaces of the friction clutch shown. If the coefficient of friction is 0.25 and the diameter of the shaft is 1.25 in., find the maximum shearing unit-stress in the shaft.

102. The stern end of a marine propeller-shaft has a diameter of 11 in., and the maximum shearing stress is 8000 lb. per sq. in. To this is connected a hollow steel tail shaft in which the maximum shearing stress is 10,000 lb. per sq. in. The internal diameter is one-half of the external diameter. Find the diameters.

Ans. $d_1 = 5.21$ in., $d_2 = 10.4$ in.

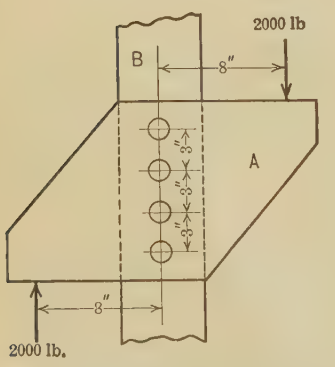


FIG. 67.—Rivets subjected to torsional shear.

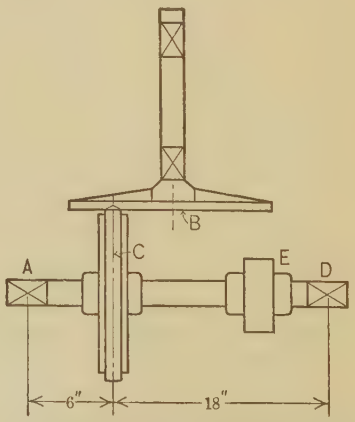


FIG. 68.—Twisting moment exerted on shaft by means of friction drive.

103. A plate A (Fig. 67) is riveted to a fixed member B by means of four $\frac{3}{4}$-in. rivets as shown in the figure. Find the shearing unit-stress at the center of each of

the rivet areas due to the 2000-lb. loads. Is the torsion formula applicable to this problem? Note that the shearing area of the rivets may be assumed to be two concentric annular areas, equivalent to the cross-sections of two concentric hollow cylinders.

104. A crown friction drive as shown in Fig. 68 is used on screw-power presses, motor trucks, etc. The cast-iron disk B rotates at 1000 r.p.m. and drives the crown wheel C, which in turn drives the chain sprocket E. The disk C is faced with leather-fiber for which the coefficient of friction is 0.35. If the diameter of the disk C is 20 in. and the normal pressure of B against C is 343 lb., what should be the diameter of the shaft between C and E in order to prevent the shearing unit-stress due to the twisting moment from exceeding 8000 lb. per sq. in. (The working stress is taken low because the load is not a steady or static load.) *Ans.* $d = 0.915$ in.

105. Compare the strengths of the following shafts when stressed to their limits of proportionality. The first is solid 21 in. in diameter and the shearing proportional limit is 25,000 lb. per sq. in.; the second is hollow, having an outside diameter of 18 in. and an inside diameter of 9 in., and a proportional limit of 45,000 lb. per sq. in.

106. How much will the allowable twisting moment of a cylindrical shaft be decreased by boring a concentric longitudinal hole through the shaft, the cross-sectional area of the hole being one-fourth of the original area of the shaft?

Ans. $6\frac{1}{4}$ per cent.

23. Twisting Moment in Terms of Horsepower and Speed.—Frequently the forces acting on the shaft are unknown, and the twisting moment must be found from the horsepower transmitted by the shaft and the speed (angular velocity) of the shaft. This may be done as follows: Let a pulley A (Fig. 69) be keyed to a shaft B and transmit a twisting moment T (equal to Pp) to the shaft when rotating at a constant angular velocity of ω radians per second or n revolutions per minute. The work done by the twisting couple T, in any angular displacement is the product of the moment of the couple and the angular displacement (in radians). Hence,

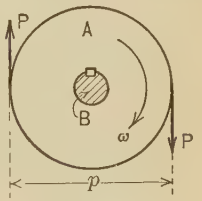


FIG. 69.

$$\text{Work done on shaft per second} = T\omega$$
$$= T\frac{2\pi n}{60}, \text{ since } \omega = \frac{2\pi n}{60} \text{ radians.}$$

If the twisting moment T is expressed in inch-pounds the work done will also be expressed in inch-pounds. But a horsepower is defined to be 550 ft.-lb. of work per sec. or 550×12 in.-lb. per sec., and hence the number of horsepowers (h.p.) transmitted by the shaft is

$$\text{(h.p.)} = \frac{T2\pi n}{60 \times 550 \times 12}, \quad \ldots\ldots \quad (23)$$

in which T is expressed in lb.-in. and n in r.p.m. Hence

$$T = \frac{60 \times 550 \times 12 \times (\text{h.p.})}{2\pi n} \text{ in.-lb.}, \quad . \quad . \quad . \quad . \quad (24)$$

which expresses the relation between twisting moment, horsepower, and speed.

PROBLEMS

107. The hydraulic turbines in the Keokuk water-power plant are rated at 10,000 h.p., with an overload capacity of 13,500 h.p. The vertical shaft connecting the turbine and the generator is 25 in. in diameter and rotates at 57.7 r.p.m. What is the maximum shearing unit-stress developed in the shaft when developing full load, and when developing maximum overload?

108. A pulley 10 ft. in diameter is mounted on a $2\frac{1}{2}$-in. shaft and rotates at 150 r.p.m. transmitting power to a belt. If the belt tensions are 1800 lb. and 1400 lb. (*a*) what horse-power is transmitted? (*b*) What is the maximum torsional shearing unit stress in the shaft? *Ans.* (*a*) (h.p.) = 57.1, (*b*) s_s = 7820 lb. per sq. in.

109. A hollow shaft is used to transmit 10,000 h.p. The outside diameter is 18 in., the speed is 90 r.p.m., and the maximum shearing unit-stress developed in the shaft is 8000 lb. per sq. in. Find the inside diameter of the shaft. *Ans.* d_1 = 12.5 in.

110. Find the diameter of the solid circular shaft required to transmit 200 h.p. at a speed of 300 r.p.m., if the allowable shearing unit-stress is 10,000 lb./in.^{2}

111. A pulley rotating at 200 r.p.m. delivers 125 h.p. to a solid steel shaft 3 in. in diameter. The pulley is prevented from rotating on the shaft by means of a steel key $\frac{1}{2}$ in. square in cross-section and 4 in. long. Find the shearing unit-stress in the key. *Ans.* 13,200 lb./in.^{2}

112. What horsepower can a solid steel shaft 6 in. in diameter transmit when rotating at 120 r.p.m. and developing a shearing unit-stress of 10,000 lb. per sq. in. in the outer fiber?

24. Angle of Twist of Cylindrical Shaft.—In some problems the size of shaft required to transmit a given twisting moment (or given horsepower at a given speed) is controlled by the allowable angle of twist rather than the allowable shearing unit-stress. In other words, stiffness rather than strength is the controlling factor in the design. The angle of twist, θ (Fig. 63), caused by a given twisting moment T may be found as follows:

As explained in Art. 6,

$$\frac{s_s}{\epsilon_s} = \text{a constant} = E_s,$$

provided that s_s does not exceed the shearing proportional limit of the material. But as shown in Arts. 21 and 22

$$s_s = \frac{Tc}{J} \quad \text{and} \quad \epsilon_s = \frac{c\theta}{l}.$$

Therefore,

$$E_s = \frac{\frac{Tc}{J}}{\frac{c\theta}{l}} = \frac{Tl}{J\theta}.$$

Hence, the angle of twist, θ, if the proportional limit of the material is not exceeded, is

$$\theta = \frac{Tl}{E_s J}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in which θ is expressed in radians, any consistent units of force and length being used to express the other quantities. Thus, T is usually expressed in inch-pounds, l in inches, E_s in pounds per square inch, and J in inches.[4]

PROBLEMS

113. A steel shaft 4 in. in diameter transmits 200 h.p. at a speed of 250 r.p.m. The distance between the driving and driven pulleys is 10 ft. Determine whether the following two requirements are satisfied:

(*a*) Maximum shearing unit-stress not to exceed 10,000 lb. per sq. in.

(*b*) Twist of shaft not to exceed 1 degree per 20 diameters of length.

Ans. $s_s = 4020$ lb. per sq. in. $\theta = 0.77°$ per 20 diam.

114. A wrought-iron shaft 7.5 ft. long and 2 in. in diameter twists through an angle of 10.5 degrees when resisting a twisting moment of 2500 lb.-ft. The shearing proportional limit of the material is 24,000 lb. per sq. in. Find the value of the shearing modulus of elasticity. *Ans.* $E_s = 9{,}370{,}000$ lb. per sq. in.

115. A steel shaft 2 in. in diameter is twisted through 1 degree in a length of L in. What is the least value L may have without causing the shearing unit-stress in the shaft to exceed 20,000 lb./in.^{2}

116. A rolled brass shaft 2 in. in diameter and a steel shaft d in. in diameter are twisted through the same angle of twist per inch of length by equal twisting moments. Find the value of d (refer to Table 1). *Ans.* $d = 1.61$ in.

117. A hollow bronze shaft having an outside diameter of 6 in. and an inside diameter of 4 in. is 12 ft. long. If the shaft is twisted by applying moments at its ends, what will be the angle of twist when the maximum shearing unit-stress is 8000 lb. per sq. in.? (Refer to Table 1.)

118. The steel shaft connecting a hydraulic turbine with an electric generator was so long that it twisted enough to cause electrical trouble in the generator. If the shaft had been replaced by one with the same diameter but made of steel having

a greater shearing strength would the amount of twist have been decreased? Which would decrease the angle of twist the more, decreasing the length a certain per cent or increasing the diameter the same per cent?

25. Shaft Couplings.—Two shafts frequently are connected by means of a bolted coupling (Fig. 70*a*) so that, as the twisting moment is transmitted from one shaft to the other, shearing stresses are developed in the bolts. For example, let the shaft A (Fig. 70*a*) exert a twisting moment on the shaft B just sufficient to overcome the resistance to motion of shaft B, the resisting moment exerted by B being equal to the moment of the shearing stresses in the bolts as indicated in Fig. 70(*b*).

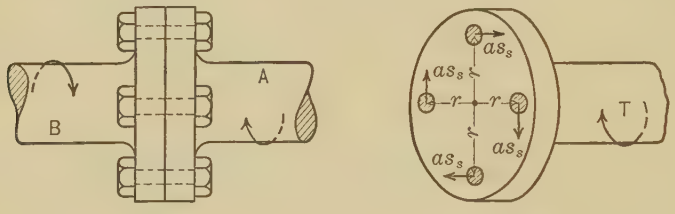


FIG. 70.—Shear in bolts of coupling.

Now if the diameter, d, of the bolts is small in comparison with the distance, r, from the center of the bolts to the center of the shaft, the shearing unit-stress, s_s, over the cross-section area, a, of each bolt may be assumed to be constant. Hence the resisting moment, T_r, is

$$T_r = m(as_s \cdot r)$$
$$= m\frac{\pi d^2}{4} s_s \cdot r,$$

in which m is the number of bolts, all of which are on a circle of radius r. And since this resisting moment is equal and opposite to the twisting moment T, then

$$T = m\frac{\pi d^2}{4} s_s \cdot r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If T is expressed in inch-pounds, d and r must be expressed in inches, and s_s in pounds per square inch. But from Art. 23

$$T = \frac{60 \times 550 \times 12 \times (\text{h.p.})}{2\pi n} \text{ in.-lb.}$$

Therefore,

$$\frac{60 \times 550 \times 12 \times (\text{h.p.})}{2\pi n} = m\frac{\pi d^2}{4} s_s \cdot r, \quad . \quad . \quad . \quad . \quad (27)$$

from which any one of the quantities may be found if all the others are known.

The resisting moment, T_r, could have been obtained from the expression $\frac{s_s J}{c}$ in which

$$J = m(\bar{J} + ar^2), \text{ see Art. 175,}$$

but $\bar{J} = \frac{1}{32}\pi d,^4$ and since d is small in comparison with r then d^2 is negligible in comparison with r^2. Hence the resisting moment is

$$T_r = \frac{s_s mar^2}{r} = m\frac{\pi d^2}{4} r s_s,$$

which is the same expression as found by the method used above.

PROBLEMS

119. A solid shaft 4 in. in diameter is to be connected to another shaft of the same size by means of a coupling as shown in Fig. 70*a*. If six $\frac{3}{4}$-in. bolts are used on a circle having a diameter of 10 in., what (h.p.) can the shaft transmit when rotating at 150 r.p.m. and when the shearing unit-stress in the bolts is 10,000 lb. per sq. in.? What will be the maximum shearing unit-stress in the shaft?

120. A coupling is to be used to connect two shafts having diameters of 4 in. The maximum allowable shearing unit-stress in the shafts is 10,000 lb. per sq. in., the diameter of the bolt circle is 8 in. and the allowable shearing unit-stress in the bolts is 9000 lb. per sq. in. Find the number of $\frac{3}{4}$-in. bolts necessary. *Ans.* Eight.

121. Two shafts, *A* and *B*, having diameters of 3 in. and 4 in., respectively, are connected by a flanged coupling. The flanges are bolted together by six bolts $\frac{3}{4}$ in. in diameter, whose centers lie on a circle having a radius of 5 in. A twisting moment of 11,000 lb.-ft. is transmitted from one shaft to the other. Which is the greater, the shearing unit-stress in shaft *A* or that in the bolts?

26. Stress beyond Proportional Limit. Modulus of Rupture.—As shown in Art. 22, the value of s_s in the torsion formula, $T = \frac{s_s J}{c}$, is the shearing unit-stress at the distance c from the axis of the shaft *only when the proportional limit of the material is not exceeded.* If the twisting moment, T, causes a unit-stress greater than the shearing proportional limit of the material the internal or resisting moment is still equal to (holds in equilibrium) the external or twisting moment, but the resisting moment is not given by the expression $\frac{s_s J}{c}$ since this expression is found by assuming that the shearing unit-stress varies directly as the distance from the center of the shaft which is true only when the stress does not exceed the proportional limit of the material.

This fact is illustrated in Fig. 71. A diameter before twisting is assumed to be a diameter (straight line) after twisting even though the material is stressed beyond the proportional limit, and hence the unit-strain in each of the two shafts (Fig. 71), varies directly as the distance from the center of the shaft. But in the shaft that is stressed beyond the proportional limit (Fig. 71*b*) the unit-stress is not proportional to the unit-strain except near the center of the shaft where the strain is small, and hence the unit-stress does not vary directly as the distance from the center of the shaft except for a short distance out from the center. If the material of the shaft is ductile enough to have a yield-

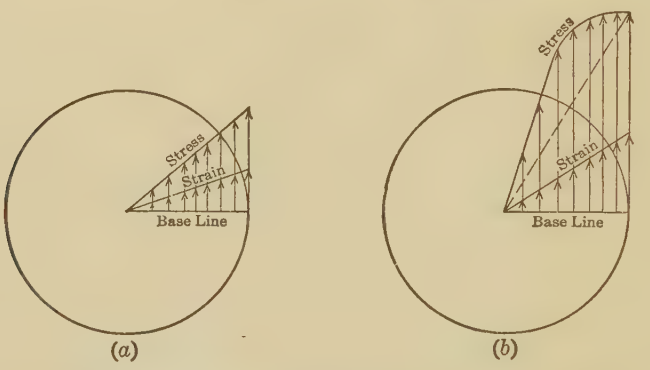


FIG. 71.—Distribution of shearing stress in shaft when stressed above proportional limit.

point the unit-stress will vary about as shown in Fig. 71(*b*) when the yield-point of the shaft is reached.

Modulus of Rupture.—Now the moment of the stresses in Fig. 71(*b*) is larger, for a given unit-stress at the surface of the shaft, than it would be if the unit-stress varied directly from zero at the center to this given value at the surface as indicated by the dotted line. Therefore, if the shearing ultimate strength [3] of the material is substituted for s_s in the torsion formula $\left(T = \frac{s_s J}{c}\right)$ the value found for T would be smaller than the maximum twisting moment that can be applied to the shaft. Tests show that the maximum twisting moment that can be applied to cylindrical steel bars is from 10 to 20 per cent greater than that calculated from the torsion formula by substituting the shearing ultimate strength for s_s.

[3] The shearing yield-point of a ductile material is regarded as its usable shearing ultimate strength.

The value of s_s found from the torsion formula by substituting for T the value of the maximum twisting moment that a shaft resists when tested to rupture is called the shearing *modulus of rupture* and will be denoted by s_r. Thus

$$s_r = \frac{T_{\max.}\, c}{J}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

It should be noted that s_r is not the unit-stress in the material caused by the moment $T_{\max.}$, and, as already noted, it is not equal to the shearing ultimate strength of the material; it is merely a value (expressed in lb. per sq. in.) from which the maximum twisting moment that a cylindrical shaft can resist, may be found.

27. Longitudinal Shearing Stress in Torsional Member.—In the preceding articles the shearing unit-stress at any point on a *transverse* section of the shaft was found. It will now be shown that at any point in the shaft there is a shearing unit-stress on a longitudinal plane equal to that on the transverse plane as indicated in Fig. 72(*c*).

Proposition.—If a shearing stress of intensity s_s occurs on a plane at a given point in a body there must exist a shearing stress of equal intensity at that point on a plane at right angles to the first plane.

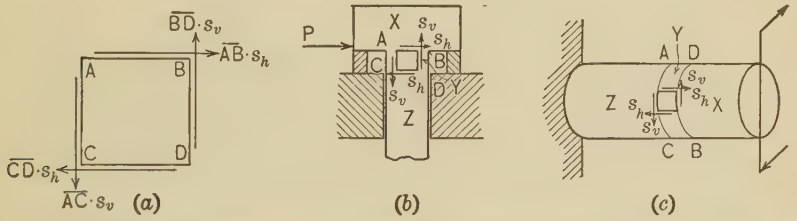


FIG. 72.—Shearing stresses of equal intensity on planes at right angles to each other.

Proof.—Let an elementary rectangular block (Fig. 72*a*) be removed from a body in which the block is subjected to shearing stresses on a pair of parallel faces, such as faces $\overline{AB}$ and $\overline{CD}$ or faces $\overline{AC}$ and $\overline{BD}$. For example, the block may be in a bolt as shown in Fig. 72(*b*), where the part X causes a shearing stress of average intensity s_h on the horizontal plane $\overline{AB}$ of part Y, and the part Z exerts a resisting stress of equal intensity on the face $\overline{CD}$. Similarly, the part X of the shaft in Fig. 72(*c*) causes an average shearing unit-stress s_v on the vertical plane $\overline{BD}$ and the part Z of the shaft exerts an equal resisting or opposite stress on the face $\overline{AC}$.

Now the shearing forces acting on a pair of faces form a couple and since the block is in equilibrium when in the body, there must be shear-

ing forces on the other pair of faces such that the moments of the two couples are equal. If the depth of the block perpendicular to the paper is assumed to be unity, then

$$\overline{AB}s_h \cdot \overline{BD} = \overline{BD}s_v \cdot \overline{AB},$$

whence

$$s_h = s_v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Now if the dimensions of the block are considered to be indefinitely small, s_h and s_v may be considered to be the stress at a point. That is, the intensities of the shearing stresses on planes at right angles to each other at any point in a body are equal; and this is true also when normal stresses act on the planes in addition to shearing stresses, as is illustrated in Fig. 15. If shearing stresses, *only*, occur on the two planes at any point in the body, the body at that point is said to be in a state of *simple shear* or of *pure shear*.

28. Diagonal Tensile (and Compressive) Stress in Torsional Member.—*Proposition.*—If a state of simple shear exists at a point in a body (as in Fig. 72*c*) there also exist normal (tensile and compressive) stresses on planes that bisect the angles between the planes on which the shearing stresses occur, and the intensities of the normal stresses are equal to those of the shearing stresses.

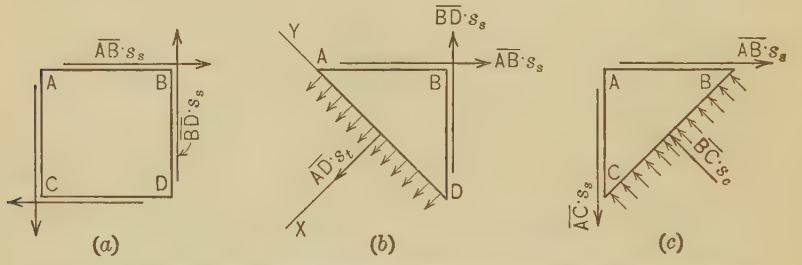


FIG. 73.—Tensile and compressive stresses resulting from simple shear.

Proof.—Let a diagonal plane $\overline{AD}$ be passed through the block of Fig. 73(*a*), or Fig. 72(*a*); the forces acting on one part of the block are shown in Fig. 73(*b*). The face of the block in Fig. 73 is assumed, for convenience, to have a thickness (perpendicular to the plane of the paper) of unity. The force on each face is equal to the area of the face times the unit-stress on that face, and since the block ABD (Fig. 73*b*) is in equilibrium, we have, by resolving the forces in the x-direction,

$$\overline{AD}s_t = \overline{AB}s_s \cos 45° + \overline{BD}s_s \cos 45° = s_s(\overline{AB} \cos 45° + \overline{BD} \cos 45°)$$

but

$$\overline{AB} \cos 45° + \overline{BD} \cos 45° = \overline{AD}.$$

Therefore,

$$s_t = s_s.$$

Similarly, from Fig. 73(*c*),

$$s_c = s_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Thus when a brittle material such as cast iron, which is weak in tension, is twisted as in Fig. 72(*c*) and Fig. 74(*a*) the material fails in tension

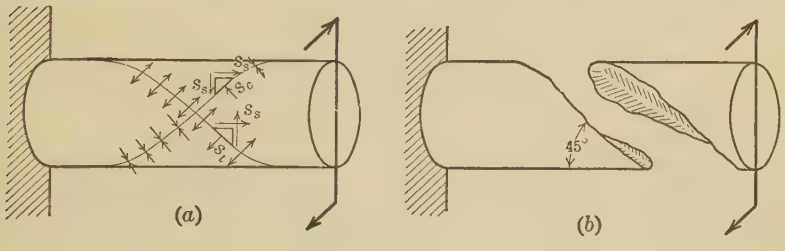


FIG. 74.—Brittle material when subjected to torsion fails in tension.

on a plane inclined (approximately 45°) to the planes on which the shearing stresses occur, as indicated in Fig. 74(*b*).[4]

PROBLEMS

122. If a cylindrical specimen of yellow pine wood 2 in. in diameter is subjected to a torsional moment of 1260 lb.-in., what is the maximum longitudinal shearing unit-stress developed in the wood? If the grain of the wood is parallel to the axis of the specimen, is the specimen likely to fail? (Consult Table 1.)

123. A cast-iron cylindrical specimen 1.5 in. in diameter is subjected to a twisting moment at each end. Calculate the maximum allowable moment that can be applied to the specimen using a factor of safety of 10 based on the ultimate strength. (Consult Table 1.) *Ans.* 1325 lb.-in.

REVIEW QUESTIONS AND PROBLEMS

124. Point out and correct the errors in the following derivation for the torsional resisting moment in a circular shaft; let it be assumed that the shearing unit-stress at any point in a transverse section varies as the distance of the point from the center of the section. Draw a sketch on which are shown da, ρ, $s_{s\rho}$, etc.

$s_{s\rho}$ = shearing unit-stress at a point at any distance, ρ, from the center of the section.

$\frac{s_{s\rho}}{\rho}$ = shearing unit-stress at the distance unity from the center.

$\frac{s_{s\rho}}{\rho} da$ = total shearing stress on the area da at the distance ρ from the center.

[4] The student may perform an experiment on a crayon of chalk to show the type of failure indicated in Fig. 74.

$\frac{s_{s\rho}}{\rho} da \cdot \rho^2$ = moment, about the center of section, of the total shearing stress on area da at the distance ρ from the center.

$$\text{Resisting moment} = T_r = \int \frac{s_{s\rho}}{\rho} \cdot \rho^2 da = s_{s\rho} \int \frac{\rho^2 da}{\rho} = s_{s\rho} \frac{J}{\rho}.$$

125. What is the justification for equating the external (twisting) moment to the internal (resisting) moment?

126. Calculate the maximum shearing unit-stress at sections AA and BB of the cylindrical shaft shown in Fig. 75 using the following data:

$T_1 = 10{,}000$ lb.-in., $T_2 = 15{,}000$ lb.-in., $d_1 = 2$ in., $d_2 = 3$ in.

Ans. $s_A = 6370$ lb. per sq. in., $s_B = 4720$ lb. per sq. in.

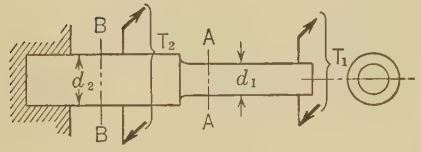


FIG. 75.

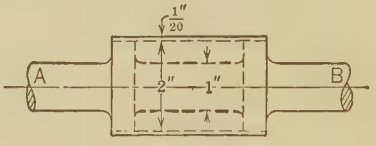


FIG. 76.

127. In Fig. 76 let the tube and shaft both be made of steel and let the tube and shaft act as one body when subjected to a twisting couple of moment T applied at A and B. If the torsional shearing unit-stress in the tube between the enlarged ends of the shaft is 8000 lb. per sq. in., what is the value of T? Use radius of tube equal 1 in. in calculating J and c. *Ans.* 3300 lb. in.

128. A shaft 2 in. in diameter transmits power at 120 r.p.m. If the maximum torsional shearing unit-stress in the shaft is 8000 lb. per sq. in., what horsepower does the shaft transmit? *Ans.* h.p. = 23.9.

129. Draw a sketch of a twisted cylindrical bar and show that the shearing unit-strain at the surface of the bar is $\epsilon_s = \frac{c\theta}{l}$.

130. From a study of the equation $\theta = \frac{Tl}{E_s J}$ answer the following question. If a steel shaft when subjected to a torsional moment T is stressed within the shearing proportional limit of the material but is twisted through too large an angle to function properly, (*a*) would the difficulty be relieved by replacing the shaft with one made of a higher-strength steel of the same dimensions? (*b*) Would increasing the diameter help? (*c*) Would increasing the length help? Explain.

Ans. (*a*) No, (*b*) yes, (*c*) no.

131. A steel tube with a wall thickness of 1/20 in. is shrunk on a steel shaft having enlarged ends as shown in Fig. 76. The shaft is held twisted by couples of 50 lb.-ft. applied at the ends of the shaft while the tube is being shrunk on. After the tube has become firmly attached to the shaft the twisting couples are removed. Calculate the twisting couple still left on the shaft. Assume that the shearing proportional limit of the material is not exceeded; use radius of tube equal 1 in. in calculating J.

Ans. 38.1 lb.-ft.

132. Calculate the shearing unit-stress in each of the rivets in Fig. 77. Each rivet has a diameter of $\frac{3}{4}$ in. *Ans.* s_A = 7980 lb. per sq. in.
s_B = 4790 lb. per sq. in.

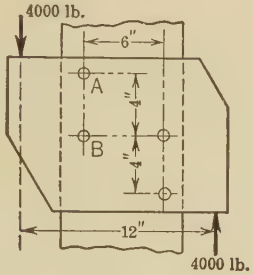


FIG. 77.

133. If in a twisted cylindrical bar a shearing unit-stress s_s exists at a point on a transverse plane, what unit-stress exists (*a*) at the same point on a longitudinal plane; (*b*) at the same point on a diagonal plane that bisects the angle between the other two planes?

CHAPTER IV

TRANSVERSE LOADS. STRESSES IN BEAMS

29. Preliminary Considerations.—A *beam* is a bar or member that is bent by forces acting perpendicular to the axis of the bar.[1] Forces that act on a beam are called *transverse* or *cross-bending* loads. It will be assumed that the cross-bending loads acting on a beam lie in one plane containing the central longitudinal axis of the beam and that this plane is a plane of symmetry of the beam.

Beams are very important members in many engineering structures and machines. Several common types of beams are described as follows:

A *simple* beam is one that rests on two supports at the ends of the beam. Fig. 78 shows a horizontal simple beam subjected to two equal concentrated loads at the third-points.

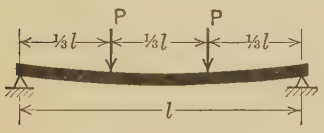


FIG. 78.—Simple beam.

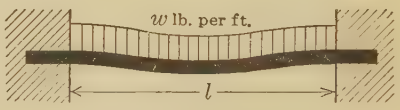


FIG. 79.—Fixed beam.

A *fixed* or *fixed-ended* beam is one so restrained at its ends that the slope of the curve of the beam at the restrained ends does not change when the load is applied. Fig. 79 shows a beam fixed at both ends, and subjected to a uniformly distributed load of w pounds per foot of length; Fig. 80 shows a beam fixed at one end and supported at the other, and subjected to a concentrated load at mid-span. The end connections of beams in structures and machines frequently offer considerable restraint, but not enough that the beams may be considered " fixed "; such beams are then intermediate between simple beams and fixed-ended beams.

A *cantilever* beam is one that is fixed at one end and free at the other. Fig. 81 shows a cantilever beam subjected to a uniformly distributed load over its entire length, and to a concentrated load at its end.

[1] If the beam acts as it is assumed to act, in this and the following chapters, its length must be at least several times its depth. Further, the thickness or width of the beam must be sufficient to prevent collapse by wrinkling or twisting; thus, extremely deep and extremely thin beams are excluded.

Fig. 82 shows an *overhanging* beam that overhangs both supports and that carries concentrated loads at the ends of the beam and a uniformly distributed load over the span between supports.

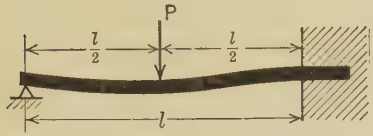


FIG. 80.—Beam fixed at one end, supported at other end.

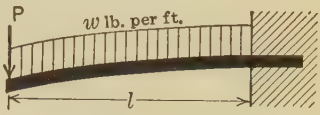


FIG. 81.—Cantilever beam.

A *continuous* beam is one that rests on more than two supports. Fig. 83 shows a continuous beam with three equal spans carrying a uniformly distributed load of w_1 pounds per foot over two spans and a uniformly distributed load of w_2 pounds per foot over the third span.

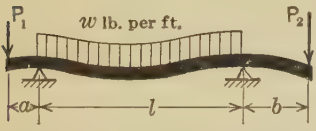


FIG. 82.—Overhanging beam.

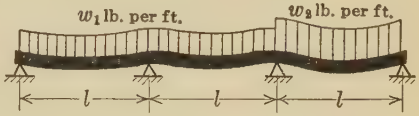


FIG. 83.—Continuous beam.

Pure bending is bending caused by couples. Thus the middle third of the beam in Fig. 78 is subjected to pure bending. Bending produced by forces that do not form couples is called *ordinary* bending. As will be shown later, a beam subjected to pure bending has no shearing stresses developed in it; the stresses on any section are normal (tensile or compressive) stresses whereas in *ordinary* bending the loads develop shearing stresses as well as normal stresses.

Reactions.—For convenience the forces exerted on a beam by the supports are called *reactions* and the other forces are called loads, but both loads and reactions are merely external forces that act on the beam and hold the beam in equilibrium.

If the loads are known, the reactions can be found by use of the equations of equilibrium *provided that there are not more than two reactions.* For, the forces that hold the beam in equilibrium constitute a parallel force system in a plane, and for such a force system there are only two independent equations of equilibrium; namely,

$$\begin{cases} \Sigma F = 0 \\ \Sigma M = 0 \end{cases} \quad \text{or} \quad \begin{cases} \Sigma M_A = 0 \\ \Sigma M_B = 0 \end{cases}$$

which state that the algebraic sum of the forces and the algebraic sum of the moments of the forces about any point are equal to zero. Or, the algebraic sum of the moments of the forces about each of two points A or B, not in a line parallel to the forces, is equal to zero.

Statically Determinate and Statically Indeterminate Beams.—Beams for which the reactions can be found from the equations of equilibrium are called *statically determinate* beams, and those for which the number of unknown reactions is greater than the number of equilibrium equations are called *statically indeterminate beams.* Simple and cantilever beams, and overhanging beams that rest on two supports, are *statically determinate* beams, whereas, fixed-ended beams and continuous beams are *statically indeterminate*, and hence require equations in addition to the equilibrium equations in order to determine the reactions.

This chapter deals mainly with the problem of determining the relation between the external forces acting on statically determinate beams and the stresses which the forces develop in the beams. Statically indeterminate beams are discussed in Chapters VI and XIII.

PROBLEMS

134. Find the reactions of the supports for the beam shown in Fig. 84; neglect the weight of the beam.

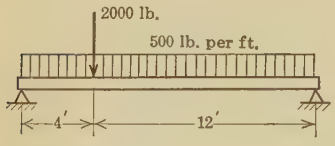


FIG. 84.

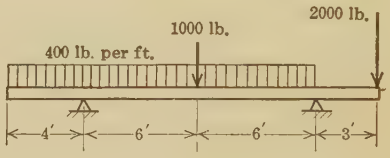


FIG. 85.

135. Find the reactions of the supports for the beam shown in Fig. 85; neglect the weight of the beam. *Ans.* $R_1 = 4267$ lb. $R_2 = 5133$ lb.

136. In Fig. 82 let $P_1 = 1000$ lb., $P_2 = 2000$ lb., $w = 200$, $a = 4$ ft., $b = 6$ ft. and $l = 20$ ft. Calculate the reactions of the supports.

137. The floor in a room supports a uniformly distributed load of 200 lb. per sq. ft. The room is 20 ft. wide and the floor rests on simple beams having supports directly beneath the walls. If the beams are spaced 6 ft. apart, what are the reactions at the ends of the beams?

138. In Fig. 81 let $P = 2000$ lb., $w = 150$, and $l = 12$ ft. Calculate the reactions (couple and vertical force) exerted by the wall on the beam.

30. Vertical Shear, Resisting Shear, Bending Moment, and Resisting Moment.—In Fig. 86 (a) is shown a simple beam subjected to two concentrated loads, the weight of the beam being negligible. The reac-

tions R_1 and R_2 are found to be 6500 lb. and 3500 lb., respectively. Let it be required to determine the character of the internal forces (stresses) that must occur on any section of the beam, such as section AA.

It is important to have a definite and correct concept of the term "stresses (internal forces) at a section of a beam" as used in this and the following articles. In visualizing these stresses it is convenient to consider the beam to be composed of fibers, a fiber being an elementary part (rod) of the beam having a cross-sectional area da and extending the length of the beam. The stresses at a section of a beam, then, are the forces exerted on the ends of these fibers at the section considered, and the force on each fiber is the product of the unit-stress on the end of the fiber and the area da of the fiber; that is, the internal force or total stress on any fiber is sda. The unit-stress s (called the fiber unit-

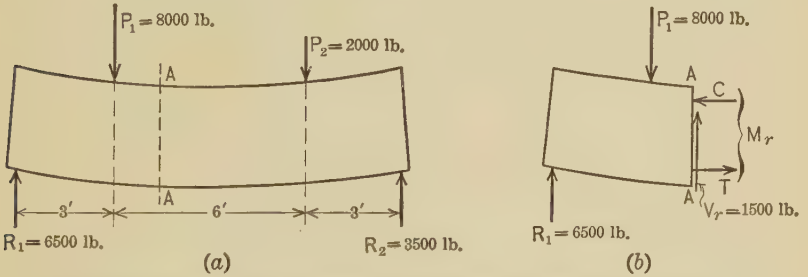


FIG. 86.—Stresses at any section hold external forces in equilibrium.

stress) varies with the position of the fiber in the beam but is assumed to be constant over the small area da of any fiber.

If the portion of the beam to the right of section AA is removed, forces equivalent to those that the right portion exerted on the left portion must be applied on section AA in order to hold the left portion in equilibrium, since it was in equilibrium before the right portion was removed. The problem, therefore, is to determine the internal forces that must act on section AA in order to hold the forces R_1 and P_1 in equilibrium.

It will now be shown that these internal forces must be equivalent to a vertical force V_r and a couple M_r composed of a compressive force C on the upper portion of the section AA and a tensile force T on the lower portion as indicated in Fig. 86(b).

The total force system acting on the portion of the beam to the left of AA must satisfy the equilibrium equations

Algebraic sum of horizontal forces = 0
Algebraic sum of vertical forces = 0
Algebraic sum of moments of forces = 0

But since it is only the unbalanced part, or resultant, of the external forces that must be held in equilibrium by the stresses on the section of the beam, it is convenient to have names and symbols for the components of the resultant, or stress-producing part, of the loads and to use them in the equations of equilibrium rather than the external forces themselves. Thus,

> The *vertical shear* for a section of a beam is the magnitude of the resultant of the forces (loads and reactions) that lie on one side of the section. Or, in other words, it is the algebraic sum of the forces that lie on one side of the section.

The symbol V will be used to denote vertical shear and, for convenience, the forces that lie to the *left* of the section will here be used.

The vertical shear is resisted by the total shearing stress on the section AA which is called the resisting shear and is denoted by V_r. Hence

> The *resisting shear* at a section of a beam is the total shearing force on the section.

Thus from the second equation above we have

$$\text{Vertical shear} = \text{Resisting shear, or } V = V_r.$$

For the beam shown in Fig. 86(a) the vertical shear V for section AA is a downward force equal to 8000 lb. − 6500 lb. or 1500 lb., and hence the resisting shear, V_r, on section AA (Fig. 86b) is an upward stress of 1500 lb. It should be noted that the vertical shear for a section does *not* act at the section: the resisting shear, however, acts at, or on, the section.

Similarly,

> The *bending moment* at a section of a beam is the moment, about the section, of the resultant of the external forces that lie on one side of the section; that is, it is the algebraic sum of the moments, about the section, of the external forces that lie on one side of the section.

The bending moment will be denoted by the symbol M and the forces that lie to the *left* of the section will here be used. The bending moment at section AA (five feet from R_1) is,

$$M = 6500 \times 5 - 8000 \times 2 = 16{,}500 \text{ lb.-ft.}$$

This bending moment about the section AA is resisted by the moment

of the tensile and compressive stresses on the section. This latter moment is called the *resisting moment.* Hence,

> The resisting moment at a section of a beam is defined to be the algebraic sum of the moments of the stresses acting on the section, about a line in the section. It will be denoted by the symbol M_r.

Thus from the third of the equilibrium equations above we have

Bending moment = Resisting moment, or $M = M_r$.

And from the first of the equilibrium equations we have

Total compressive stress = total tensile stress, or $T = C$.

Summarizing: since any portion of the beam (Fig. 86*b*) is in equilibrium the forces acting on the portion must satisfy the equations

Algebraic sum of horizontal forces = 0, or $T = C$ . . . . (31)
Algebraic sum of vertical forces = 0, or $V = V_r$ . . . . (32)
Algebraic sum of moments of forces = 0, or $M = M_r$ . . . . (33)

31. The Flexure Formula.—By expressing M_r at any section of the beam in terms of the unit-stress on any fiber and the dimensions of the section, and then substituting this expression in eq. (33), the flexure formula is obtained. This formula gives the desired relation between the external forces acting on the beam and the normal unit-stress at any point on a particular section of the beam.

In order to find the expression for M_r at any section, the way the fiber unit-stress varies with the position of the fiber in the beam must be known. It is found (see page 84) that on one line in the section the stress is zero; this line is called the *neutral axis.* Further, it is found that

> The normal (tensile or compressive) unit-stress on a fiber at any section of the beam is directly proportional to the distance of the fiber from the neutral axis.

Thus, if s_y and s (Fig. 87) denote the unit-stresses on fibers at the distances y and c, respectively, from the neutral axis, the above statement is expressed mathematically as follows:

$$\frac{s_y}{s} = \frac{y}{c} \quad \text{or} \quad \frac{s_y}{y} = \frac{s}{c} = \text{a constant.} \quad . \quad . \quad . \quad . \quad (34)$$

Expression for Resisting Moment.—An expression may now be found

for the resisting moment (the algebraic sum of the moments of the stresses on the cross-section about the neutral axis) in terms of the unit-stress on any fiber and the dimensions of the cross-section of the beam as follows:

The unit-stress, s_y, on a fiber at the distance y from the neutral axis (Fig. 87) may be assumed to be constant over the cross-sectional area, da, of the fiber, and hence,

Total stress on area of one fiber $= s_y da$.

Moment of total stress on one fiber $= s_y y da$.

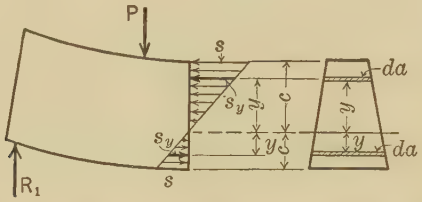


FIG. 87.—Intensity of stress varies directly as distance from neutral axis.

Sum of moments of stresses on all fibers $= M_r = \int s_y y da.$

This may be written $M_r = \int \frac{s_y}{y} y^2 da$, and since as stated above, $\frac{s_y}{y} = \frac{s}{c} = a\ constant$, we have

$$M_r = \frac{s}{c}\int y^2 da = \frac{sI}{c},$$

in which $I = \int y^2 da$ is the moment of inertia of the cross-section of the beam with respect to the neutral axis, since y is measured from the neutral axis (see Appendix I).

But equilibrium requires that $M = M_r$. Therefore,

$$M = \frac{sI}{c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

in which s is the normal unit-stress on any fiber at the distance c from the neutral axis, but since the maximum value of s is usually desired, the distance c will usually be taken as the distance to the most remote fiber. This formula is known as the *flexure formula*. If s is expressed in pounds per square inch as is customary in the United States then M must be expressed in pound-inches, I in inches to the fourth power, and

c in inches. Further, $\frac{I}{c}$ is called the *section modulus* of the beam and is expressed in inches^3. The values of the section modulus for steel rolled sections are given in Appendix III. If Z is used to denote the section modulus, the flexure formula may be written

$$s = \frac{M}{Z}. \quad \ldots \quad (36)$$

Position of the Neutral Axis.—The value of I in the flexure formula cannot be found unless the position of the neutral axis in the area is known. Now, as noted in Art. 30, the sum of all the horizontal stresses at any section must be equal to zero; this condition serves to locate the neutral axis. Thus, from Fig. 87 we have

$$\int s_y da = 0 \quad \text{or} \quad \int \frac{s_y}{y} y da = 0,$$

and hence

$$\frac{s_y}{y} \int y da = 0, \text{ since } \frac{s_y}{y} = \text{a const.}$$

But $\frac{s_y}{y}$ is not equal to zero. Therefore, $\int y da = 0$, in which y is measured from the neutral axis; but $\int y da$ is, by definition, the moment of the cross-sectional area of the beam with respect to the neutral axis (see Fig. 88), and may be written $a\bar{y}$, in which a is the total area of the cross-section and $\bar{y}$ is the distance of the centroidal axis from the neutral axis. Thus,

$$\int y da = a\bar{y} = 0.$$

But a is not equal to zero. Therefore,

$$\bar{y} = 0,$$

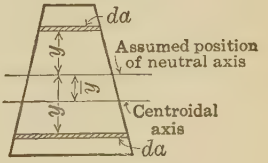


FIG. 88.

which states that the distance from the centroidal axis to the neutral axis is zero, and hence the neutral axis is coincident with the centroidal axis. It is assumed that the student is familiar with the methods of locating the centroids of areas; this topic, however, is treated in Appendix I. Further, the methods for determining the moment of inertia of an area with respect to an axis are also discussed in Appendix I.

Distribution of Stress on Section.—In the above derivation of the flexure formula the way the unit-stress was distributed over the area

had to be known. To determine this distribution there is required a knowledge of:

(*a*) The way in which the strain of a fiber varies with the position of the fiber in the beam when the beam is bent, and

(*b*) The relation, for the material of which the beam is made, of the unit-stress in a fiber to the unit-strain of the fiber.

Information concerning these points comes mainly from the results of experiments:

(*a*) When a simple horizontal beam is bent the fibers on the top side shorten and those on the bottom elongate, and the fibers in one plane within the beam do not deform. This plane is called the *neutral surface*, and the line of intersection of the neutral surface and a cross-section of the beam is called the *neutral axis* for the section. Further, when a beam is subjected to bending, experiments show that if two straight lines, *DE* and *FG* (Fig. 89) are drawn, before the beam is bent, on the side of the beam a short distance, *MN*, apart, these lines will still be approximately straight lines, *D'E'* and *F'G'*, after the beam is bent. Thus, one fiber *MN*, in the neutral surface, remains constant in length and

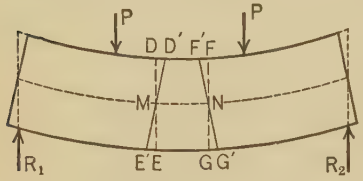


FIG. 89.—Strain of fibers in a beam.

> The strain of any fiber is directly proportional to the distance of the fiber from the neutral surface. Further, since the original lengths of all the fibers are equal the unit-strain of any fiber is also proportional to the distance of the fiber from the neutral surface.

(*b*) Now if one of the fibers were removed from the beam and subjected to an axial load in a testing machine so that the unit-stress $\left(\text{equal to } \frac{P}{a}\right)$ and the unit-strain $\left(\text{equal to } \frac{e}{l}\right)$ could be measured it would be found, as stated in Art. 6, that the *unit-stress* in the fiber is proportional to the unit-strain of the fiber *provided that the proportional limit of the material is not exceeded.* Further, it is assumed that the fiber when in the beam acts according to the same law as when tested alone.

Therefore, since the unit-strain of a fiber is directly proportional to the distance of the fiber from the neutral surface, and since the unit-stress on a fiber is directly proportional to the unit-strain of the fiber, it follows that

The unit-stress on a fiber of a beam at any section of the beam is directly proportional to the distance of the fiber from the neutral axis.

ILLUSTRATIVE PROBLEMS

Problem 139.—A simple beam having a rectangular cross-section (Fig. 90) is subjected to a uniformly distributed load of 400 lb. per ft. (including the weight of the beam) over the whole span, and a concentrated load of 2000 lb. at a distance of 4 ft. from the left support. Find the tensile unit-stress on the outer fiber of the beam at the section AB, 5 ft. from the left support.

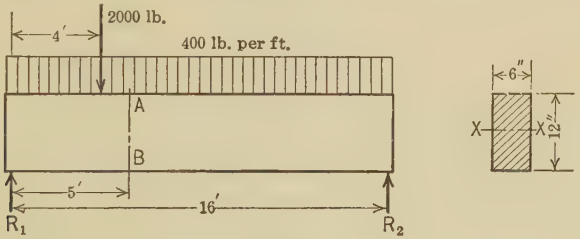


FIG. 90.

Solution.—First Method.—The reactions are found to be

$$R_1 = 4700 \text{ lb.}, \quad R_2 = 3700 \text{ lb.}$$

The bending moment M at the section AB is

$$\begin{aligned} M &= 4700 \times 5 - 400 \times 5 \times 2.5 - 2000 \times 1 \\ &= 16{,}500 \text{ lb.-ft.} = 198{,}000 \text{ lb.-in.} \end{aligned}$$

The centroidal axis (and hence the neutral axis) of the cross-section is the central horizontal axis, XX, and the moment of inertia of the cross-section about the neutral axis is (see Art. 171)

$$\bar{I} = \tfrac{1}{12}bd^3 = \tfrac{1}{12}6(12)^3 = 864 \text{ in.}^4$$

The tensile unit-stress on the bottom fiber at the section AB, then, is

$$s = \frac{Mc}{I} = \frac{198{,}000 \times 6}{864} = 1370 \text{ lb. per sq. in.}$$

provided that this unit-stress is not greater than the proportional limit of the material. A compressive unit-stress of the same magnitude occurs on the top fiber.

Second Method of Solution.—Instead of expressing the resisting moment as the algebraic sum of the moments of the stresses on the fibers of the cross-section, which leads to the expression sI/c, the resisting moment may be expressed as the moment of the couple formed by the resultants of the tensile and compressive stresses. Thus as shown in Fig. 91 the action line of the resultant, C, of the compressive stresses is $\frac{2}{3}$ of $\overline{OA}$ from O, and similarly the resultant T acts at a distance of $\frac{2}{3}$ $\overline{OB}$ from O. Hence the resisting moment is $C \times \frac{2}{3}d$ or $T \times \frac{2}{3}d$. Further, the magnitude of C

(and T) is the product of the average unit-stress, $\frac{1}{2}s$, on the area above (or below) the neutral axis and that area. Thus

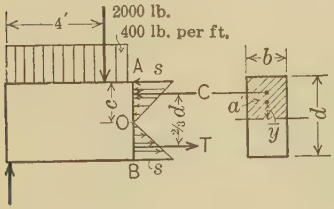


FIG. 91.—Resisting couple.

$$C = T = \tfrac{1}{2}s \times \tfrac{1}{2}bd,$$

and this may be written

$$C = T = \frac{s}{c} \times \frac{1}{8}\, bd^2.$$

Therefore the resisting moment

$$M_r = \frac{s}{c} \times \frac{1}{8}\, bd^2 \times \frac{2}{3}\, d = \frac{s}{c}\, \frac{bd^3}{12},$$

and hence,

$$M = \frac{s}{c}\, \frac{bd^3}{12} = \frac{sI}{c},$$

which is the same equation as used in the first method of solution. It is important to note that the resultants T and C do not act at a distance of $\frac{2}{3}$ of c from O unless the section of the beam is of constant width.

Problem 140.—A T-beam (Fig. 92) is subjected to a concentrated load of 4000 lb. at the center of the span. The beam is made of material having a tensile proportional limit of 4000 lb. per sq. in. and a compressive proportional limit of 8000 lb. per sq. in. Find the maximum tensile and compressive unit-stresses at the mid-

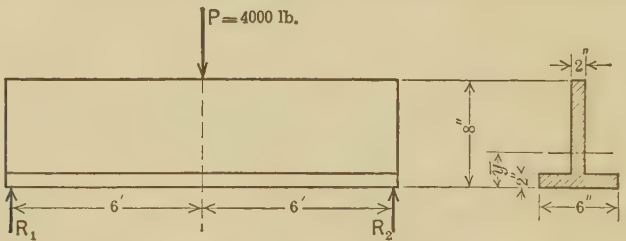


FIG. 92.—T-beam; used for material that is weak in tension.

span section, and find the factors of safety of these stresses based on the proportional limits.

Solution.—The distance, y, of the centroidal axis (and hence of the neutral axis) from the bottom line of the section is found to be 3 in. Thus (see Art. 165),

$$a\bar{y} = \Sigma a_0 y_0$$

$$\bar{y} = \frac{12 \times 5 + 12 \times 1}{12 + 12} = 3 \text{ in.}$$

The moment of inertia of the cross-section with respect to the neutral axis is found as follows (see Art. 172 for another solution):

$$\bar{I}_x = \tfrac{1}{12}\, 2(6)^3 + 12 \times (2)^2 + \tfrac{1}{12}\, 6(2)^3 + 12 \times (2)^2 = 136 \text{ in.}^4$$

The bending moment about the mid-section is

$$M = R_1 \times \frac{l}{2} = 2000 \times 72 = 144{,}000 \text{ lb.-in.}$$

Therefore, the maximum tensile unit-stress (on the bottom fiber of the beam) at the mid-section is

$$s_t = \frac{Mc}{I} = \frac{144{,}000 \times 3}{136} = 3180 \text{ lb./sq. in.}$$

and the maximum compressive unit-stress (on the top fiber) is

$$s_c = \frac{5}{3} s_t = \frac{5}{3} 3180 = 5300 \text{ lb. per sq. in.}$$

The factors of safety for the tensile and compressive stress, respectively, are

$$f_t = \frac{4000}{3180} = 1.26, \quad f_c = \frac{8000}{5300} = 1.51$$

Thus the tensile stress although smaller than the compressive stress has the smaller factor of safety. This problem shows the advantage of a T-section (or a similar section) for material (such as cast iron) that is not equally strong in tension and compression.

PROBLEMS

141. Find the fiber unit-stress at a point on the section AB (Fig. 90) at a distance of 2 in. from the top face of the beam. *Ans.* $s = 913$ lb. per sq. in.

142. Find the unit-stress on the bottom fiber of the beam in Fig. 90 on a section beneath the concentrated load.

143. Find the maximum compressive unit-stress on a section of the beam in Fig. 92 at a distance of 4 ft. from the left end.

144. If each load P in Fig. 78 is 1200 lb. and l is 16 ft., and the cross-section of the beam is a rectangle 4 in. wide by 12 in. deep, what is the maximum tensile and compressive unit-stress at any section between the loads?

145. A steel shaft 4 in. in diameter is used as a cantilever beam and loaded as shown in Fig. 93. Find the maximum unit-stress in the section, AB, at the wall. *Ans.* $s = 6880$ lb. per sq. in.

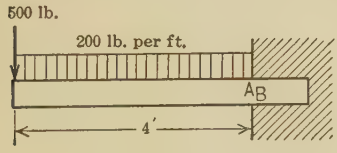


FIG. 93.

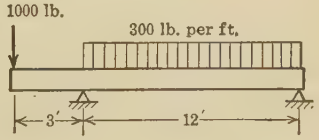


FIG. 94.

146. Find the maximum tensile fiber unit-stress on the section above the left support of the beam shown in Fig. 94 whose cross-section is composed of two 4 in., 5.4 lb. channel sections placed back to back. Find also the maximum tensile fiber unit-stress on a section midway between the supports. State in each case whether the fiber on which the unit-stress occurs is on the top or bottom of the beam. (Consult Table B in App. III.)

32. Section of Maximum Bending Moment.—If a beam has a constant cross-section throughout its length, the values of $\frac{I}{c}$ (section modulus) for all sections are equal, and hence the maximum value of s in the flexure formula, $M = \frac{sI}{c}$, occurs in the section at which the bending moment, M, is a maximum. It is important, therefore, to obtain a method of locating the section at which the bending moment is a maximum. This section is called the *dangerous section* of the beam, and may be located in accordance with the following statement:

> The section of a beam at which the bending moment is maximum is the section at which the vertical shear is either equal to zero or changes sign.

Proof.—In Fig. 95 is represented a beam subjected to a concentrated and a uniformly distributed load. The bending moment, M_x, at the distance x from the left support is

$$M_x = R_1 x - P(x - a) - \frac{wx^2}{2}.$$

Now the value of x that will make M_x a maximum is the value that will make the first derivative of M_x with respect to x equal to zero. The first derivative of M_x with respect to x is

$$\frac{dM_x}{dx} = R_1 - P - wx.$$

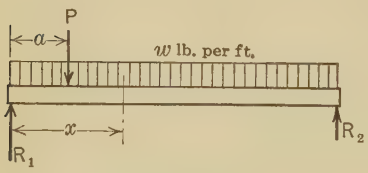


FIG. 95.

Therefore, the value of x that will make M_x a maximum may be found from the equation

$$R_1 - P - wx = 0.$$

But $R_1 - P - wx$ is the vertical shear for the section at the distance x from the left support. Therefore, the section at which the moment is maximum is the section for which the vertical shear is zero.

A convenient way of locating the section for which the vertical shear is zero (dangerous section) is to draw a shear diagram as discussed in the following article.

Further, the relation between the bending moment and the vertical shear found above, namely $V = \frac{dM}{dx}$, is of great importance; it will be

discussed in greater detail in connection with shear and moment diagrams in the next article.

An alternative method of deriving this equation is as follows: In Fig. 96 is shown a part of a beam included between two sections one being at the distance x (Fig. 95) from the left end of the beam and the other at the distance $x + dx$. All the forces acting on this element of the beam are shown in Fig. 96; the moment and shear on the first section are M and V, and on the second section $M + dM$ and $V - dV$, and the load is wdx. These forces hold the element in equilibrium, hence the algebraic sum of their moments is equal to zero. Thus taking moments about O, we have

$$M + Vdx - wdx \times \tfrac{1}{2}dx - (M + dM) = 0,$$

and since products of differentials may be neglected, this equation reduces to

$$Vdx = dM \quad \text{or} \quad V = \frac{dM}{dx}. \quad . \quad (37)$$

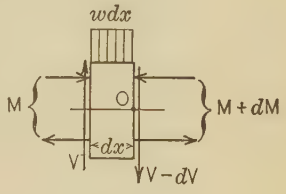


FIG. 96.

The use of this equation for determining bending moments from a shear diagram is discussed in Art. 34.

33. Shear and Moment Diagrams.—*Shear Diagram.*—A shear diagram for a beam is a curve in which the abscissas represent distances along the beam and the ordinates represent the vertical shears for the sections at which the ordinates are drawn.

For example let it be required to draw a shear diagram for the beam shown in Fig. 97(*a*). The reactions are found to be

$$R_1 = 7000 \text{ lb.} \quad \text{and} \quad R_2 = 5000 \text{ lb.}$$

Now the vertical shear for section A (just to the right of the left support) is $V_A = R_1 = 7000$ lb. upward; it is represented by AE in Fig. 97(*b*), an upward shear being considered positive and plotted above the base line.

The vertical shear for section C (just to the left of the load P) is

$$V_C = 7000 - 4 \times 500 = +\ 5000 \text{ lb.}$$

and the vertical shear for any section between A and C at the distance x, say, from A is

$$V_x = 7000 - 500x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and hence the vertical shear decreases with x at the constant rate of 500 lb. per foot of length of the beam from 7000 lb. at A ($x = 0$) to 5000 lb. at C ($x = 4$) as shown in Fig. 97(*b*).

The vertical shear at section D (just to the right of the concentrated load) is

$$V_D = 7000 - 500 \times 4 - 4000 = +\ 1000 \text{ lb.}$$

Thus, there is an abrupt change in the vertical shear in passing from section C to D due to the concentrated load, but the shear does not (in

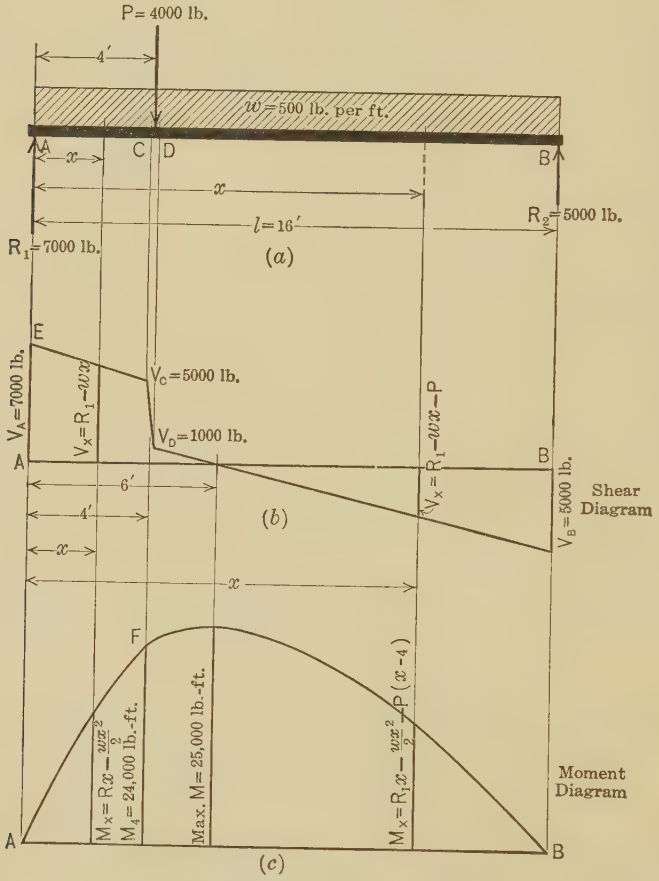


FIG. 97.—Shear and moment diagrams.

this particular problem) pass through zero under the load; the shear must, therefore, be equal to zero for some section to the *right* of the concentrated load. Now, the vertical shear for any section between D and B at a distance x from the left support is

$$V_x = 7000 - 4000 - 500x, \quad . \ . \ . \ . \ . \quad (39)$$

and hence the vertical shear decreases at the constant rate of 500 lb. per foot of length of the beam from 1000 lb. at D to $-$ 5000 lb. for a section just to the left of the right reaction. At 6 feet from the left support the vertical shear is zero and hence the bending moment at this section is maximum.

It is important to note that if a concentrated load acts on a beam in addition to a distributed load the shear may pass through zero under the load, and if it does the location of the dangerous section cannot be found by equating any expression such as equations (38) and (39) to zero and solving for x. It is desirable, therefore, in finding the dangerous section, to plot a shear diagram. If a beam is acted on by concentrated loads only, the dangerous sections will *always* occur under one of the loads since the shear changes only at the sections where the loads act.

Moment Diagram.—A *moment diagram* for a beam is a curve in which the abscissas represent distances along the beam and the ordinates represent the bending moments at the sections at which the ordinates are drawn.

Let it be required to draw a moment diagram for the beam shown in Fig. 97(a). The bending moment at the section A is equal to zero since the moment arm of R_1 about A is zero. The bending moment at any section between A and C, at a distance x from the left support, is

$$M_x = R_1 x - \frac{wx^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in which x cannot have a value greater than 4. The bending moment at any section between D and B (x greater than 4), at a distance x from the left support, is

$$M_x = R_1 x - \frac{wx^2}{2} - P(x - 4), \quad . \quad . \quad . \quad . \quad . \quad (41)$$

in which x cannot have a value less than 4. The value of the bending moment at the section beneath the load ($x = 4$) may be found from either equation by making x equal to 4.

If various values of x from 0 to 4 be substituted in equation (40) and the resulting values of the bending moments plotted, the moment diagram obtained is AF (Fig. 97c), and if values of x from 4 to 16 be substituted in equation (41) the resulting moment diagram is FB (Fig. 97c).

It will be noted that the maximum ordinate to the moment curve occurs at the section for which the shear is zero. Further, it is important to observe that the moment diagram is composed of two *distinct* curves, AF and FB, which have only one point, F, in common, and that the

maximum bending moment can be found from the equation of only one of these curves. Thus in the beam of Fig. 97(*a*) the value of x for which the bending moment is maximum cannot be substituted in equation (40), since 4 is the greatest value x can have in this equation. If values of x greater than 4 are substituted in equation (40) the corresponding values of M_x will have no physical meaning; and if values of x less than 4 are substituted in equation (41) the corresponding values of M_x will have no physical meaning.

Sign of Bending Moment.—The bending moment at any section of a horizontal beam is considered to be positive when it produces compressive stress on the top fibers of the beam at the section and tensile stress on the bottom fibers. Thus, if the bending moment is obtained from the forces that lie to the left of the section the bending moment is positive when it is clockwise, and if obtained from the forces to the right of the section the bending moment is positive when it is counterclockwise.

ILLUSTRATIVE PROBLEMS

Problem 147.—A solid cylindrical steel shaft is used as a simple beam over a span of 12 ft. and is loaded as shown in Fig. 98(*a*). Draw the shear and moment diagrams and find the diameter of the shaft if the working fiber unit-stress is 16,000 lb. per sq. in.

Solution.—The reactions are found to be

$$R_1 = 6883 \text{ lb.} \quad \text{and} \quad R_2 = 5717 \text{ lb.}$$

The shear and moment diagrams are shown in Fig. 98(*b*). The dangerous section is under the concentrated load and the maximum moment is

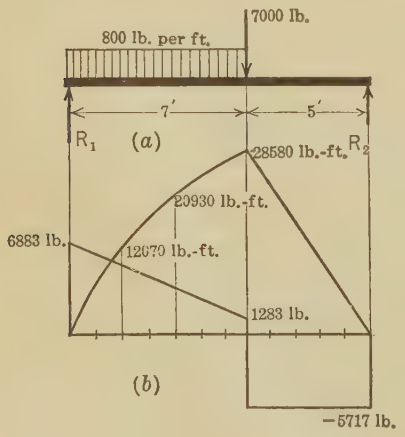


FIG. 98.

$$M = 6883 \times 7 - 800 \times 7 \times \frac{7}{2}$$

$$= 28{,}580 \text{ lb.-ft.}$$

The maximum fiber unit-stress is

$$s = \frac{Mc}{I} = \frac{28{,}580 \times 12 \times \frac{d}{2}}{\frac{\pi d^4}{64}}.$$

Hence,

$$d^3 = \frac{28{,}580 \times 12 \times 32}{16{,}000 \times \pi} = 218 \text{ in.}^3$$

whence

$$d = 6.02 \text{ in.}$$

Problem 148. Show that the maximum bending moment for a simple beam loaded as shown in Fig. 99(*a*) is

$$M = \tfrac{1}{6} Wl$$

in which W is the total load on the beam and l is the span of the beam.

Solution.—From symmetry it is known that the shear is zero at the center of the span and that each reaction equals $W/2$. Let the intensity of the load at the center be w_c lb. per ft. Now the resultant, R, of the load to the left of the mid-span section is equal to the average load per foot ($w_c/2$) times the length ($l/2$). But the action line of R is at the distance of $\frac{2}{3} \times l/2$ from the left support. Thus the distributed load to the left of the center of the beam may be replaced by R as shown in Fig. 99(*b*) in getting the moment about the mid-span section. The bending moment about the mid-span section then, is

$$M = R_1 \times \frac{l}{2} - R \frac{1}{3} \frac{l}{2}$$

$$= \frac{1}{2} W \frac{l}{2} - \frac{w_c}{2} \frac{l}{2} \times \frac{1}{3} \frac{l}{2}$$

But, the average load per foot $\left(\frac{w_c}{2}\right)$ times the span length (l) equals W, $\left(W = \frac{w_c}{2} l\right)$.

Hence,

$$M = W \frac{l}{4} - \frac{Wl}{12} = \frac{1}{6} Wl.$$

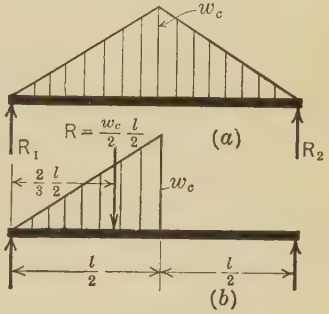


FIG. 99.

PROBLEMS FOR ARTICLES 31 TO 33

149 to **154.** Verify the shear and moment diagrams shown in Figs. 100 to 105; also verify the expressions for the vertical shear and the bending moment at a section at the distance x from the left end of the beam, and the expressions for the maximum moment, as given in Figs. 100 to 105.

Prob. 149

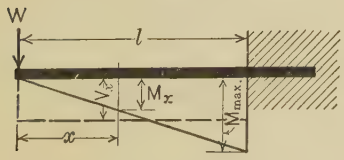


FIG. 100.

$V_x = -W$, $V_{max.} = -W$.

$M_x = -Wx$, $M_{max.} = -Wl$.

Prob. 150

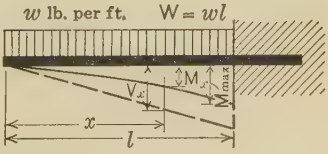


FIG. 101.

$V_x = -wx$, $V_{max.} = -wl = -W$.

$M_x = -\frac{wx^2}{2}$, $M_{max.} = -\frac{wl^2}{2} = -\frac{Wl}{2}$.

Prob. 151

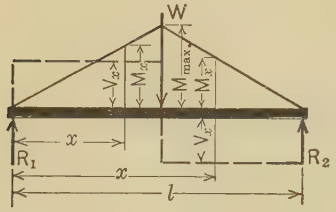


FIG. 102.

$V_x = \frac{W}{2}$, when $x = 0$ to $\frac{l}{2}$.

$V_x = \frac{W}{2} - W = -\frac{W}{2}$, when

$x = \frac{l}{2}$ to l.

$V_{max.} = \pm \frac{W}{2}$.

$M_x = \frac{W}{2} x$, when $x = 0$ to $\frac{l}{2}$.

$M_x = \frac{W}{2} x - W\left(x - \frac{l}{2}\right)$, when

$x = \frac{l}{2}$ to l.

$M_{max.} = \frac{W}{2} \cdot \frac{l}{2} = \frac{1}{4} Wl$.

Prob. 152

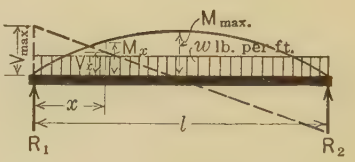


FIG. 103.

$V_x = \frac{W}{2} - wx$, $W = wl$.

$V_{max.} = \pm \frac{W}{2}$.

$M_x = \frac{W}{2} x - \frac{wx^2}{2}$,

$M_{max.} = \frac{W}{2} \cdot \frac{l}{2} - \frac{W}{2} \frac{l}{4} = \frac{1}{8} Wl = \frac{1}{8} wl^2$.

Prob. 153

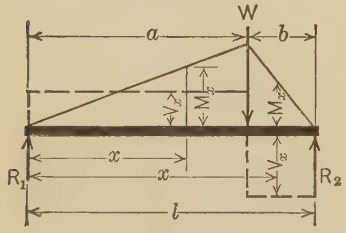


FIG. 104.

$R_1 = \frac{Wb}{l}$, $R_2 = \frac{Wa}{l}$.

For $x = 0$ to a:

$V_x = \frac{Wb}{l}$ and $M_x = \frac{Wb}{l} x$

Prob. 154

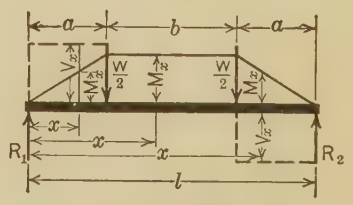


FIG. 105.

For $x = 0$ to a:

$V_x = \frac{W}{2}$, and $M_x = \frac{W}{2} x$.

For $x = a$ to $(a + b)$:

$V_x = 0$, and $M_x = \frac{W}{2} a$,

Prob. 153—*Cont.*

For $x = a$ to l:

$$V_x = -\frac{Wa}{l} \text{ and } M_x = \frac{Wb}{l}x - W(x-a)$$

$$V_{\max.} = -\frac{Wa}{l}, M_{\max.} = \frac{Wb}{l}a.$$

Prob. 154—*Cont.*

For $x = (a+b)$ to l:

$$V_x = -\frac{W}{2}, \text{ and}$$

$$M_x = Wa - \frac{W}{2}(x-b)$$

$$M_{\max.} = \frac{W}{2}a.$$

155. Plot a shear diagram and find the dangerous section for the beam shown in Fig. 106. Find also the maximum bending moment and the maximum fiber unit-stress in the beam.

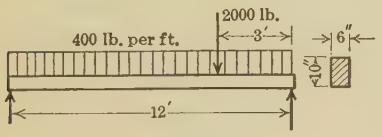


FIG. 106.

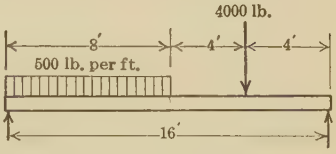


FIG. 107.

156. Draw to scale a shear and a moment diagram for the beam shown in Fig. 107. If the beam is a cylindrical bar what diameter should it have if the working fiber stress is 16,000 lb. per sq. in.? *Ans.* $d = 4.97$ in.

157. A simple beam having a span of 12 ft. is made of yellow pine for which the working fiber stress is 1000 lb. per sq. in. The beam carries concentrated loads of 2000, 4000 and 6000 lb. at distances of 3, 6 and 7 ft., respectively, from the left support and also a total uniformly distributed load of 1200 lb. If the width of the beam is 10 in. what should be the depth? *Ans.* $d = 15.2$ in.

158. A freight-car axle (Fig. 108) is 4 in. in diameter. The distance from the center of an axle box to the action line of the rail pressure is 8 in. and the distance between the rail pressures is 4.90 ft. Find the maximum fiber unit-stress in the axle if each rail pressure is 5 tons.

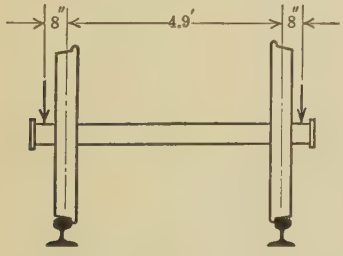


FIG. 108.

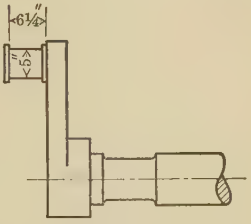


FIG. 109.

159. The steel crankpin (Fig. 109) of a steam engine has a force of 25,000 lb. transmitted to it by the connecting rod. The dimensions of the pin shown were determined from the allowable bearing pressure. Find the allowable bearing pressure, and the maximum fiber unit-stress in the pin. The load may be assumed to be uniformly distributed over the length of the pin.

Ans. $s_b = 800$ lb. per sq. in., $s = 6370$ lb. per sq. in.

160. A simple beam having a span of 16 ft. carries a total uniformly distributed load of W lb., and a concentrated load, equal to $W/2$, at the center of the span. If the cross-section of the beam is rectangular, 6 in. wide by 12 in. deep, and the allowable fiber unit-stress is 800 lb. per sq. in., find the value of W.

161. The cast-iron frame shown in Fig. 110 is subjected to a load P of 4000 lb. Find the maximum tensile and compressive fiber stresses at the section AB. The dimensions of the area at the section AB are shown in the figure.

Ans. $\bar{y}_1 = 3.5$ in., $\bar{I} = 391$ in.^{4}, $s_t = 858$ lb. per sq. in., $s_c = 1590$ lb. per sq. in.

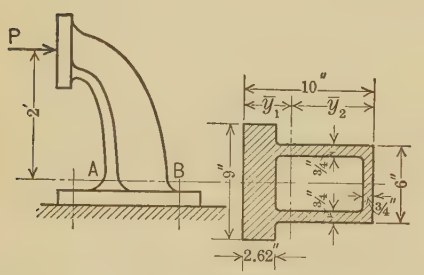


FIG. 110.

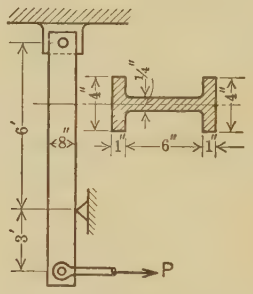


FIG. 111.

162. What force P (Fig. 111) will cause a maximum fiber unit-stress of 16,000 lb. per sq. in. in the I-beam? The dimensions of the cross-section are shown in the figure.

163. The load P (Fig. 112) is 1000 lb. What should be the depth d of the timber beam A if the working unit-stress in the material is 1200 lb. per sq. in.?

Ans. $d = 14.5$ in.

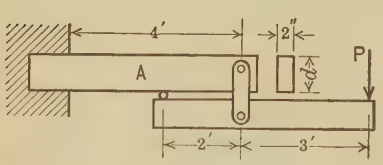


FIG. 112.

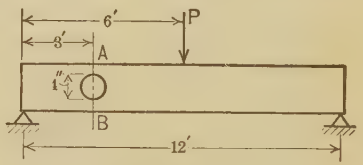


FIG. 113.

164. The simple timber beam shown in Fig. 113 has a hole 4 in. in diameter bored through the beam, the center line of the hole lying in the neutral surface and in the section AB, 3 ft. from the left support. The beam is subjected to a load P of 5000 lb. Find the maximum unit-stress that occurs in the section AB and also in the section for which the vertical shear changes sign. The beam has a rectangular section 4 in. wide and 12 in. deep.

165. Sand is piled on a floor so that the load on the beams supporting the floor varies from zero pressure at one end to a pressure of w lb. per ft. at the other end (Fig. 114). Show that the vertical shear is zero for a section at a distance of $0.577l$ from the left support, and that the maximum bending moment is $0.0642\ wl^2$.

166. Draw the shear and moment diagrams for the beam shown in Fig. 115. Find the maximum unit-stress in the beam if the beam has a rectangular cross-section 4 in. wide by 8 in. deep. *Ans.* $s = 984$ lb. per sq. in.

FIG. 114.—Non-uniformly distributed load.

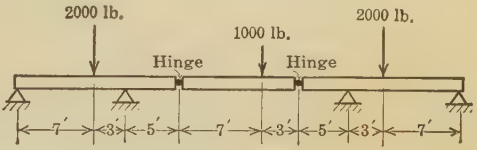


FIG. 115.

167. A simple beam 12 ft. long supported at its ends carries a uniform load of 800 lb. per ft. of length over the 6 ft. lying between points 2 ft. and 8 ft., respectively, from the left end of the beam. Draw the shear and moment diagrams and give the equation for each part of the moment curve.

168. A hollow rectangular beam having a span of 16 ft. rests on simple supports and carries a uniformly distributed load of 500 lb. per ft. of length on one-half of the span and a concentrated load of 4000 lb. in the middle of the other half. The beam is 6 in. wide and 10 in. deep. The sides and top are 1 in. thick and the bottom is 3 in. thick. Compute the maximum bending unit-stress in the beam. *Ans.* 2810 lb./in.^{2}

169. A beam 15 ft. long is partially restrained at the left end and simply supported at the right end. It carries a uniform load of 160 lb. per ft. over the entire span. There is a negative bending moment of 3000 lb.-ft. acting at the left end of the beam. Draw the shear and moment diagrams and find the maximum bending moment in the beam.

170 to **174.** The following problems are to be solved with the aid of a steel company's handbook, or the tables in Appendix III. Find the missing terms in the table. All the beams are simple beams and all distances are measured from the left support. Use a working stress of 16,000 lb./in.^{2}

Prob. No.	Length of Span (feet)	Uniform Load (lb./ft.)	Concentrated Load (lb.)	Distance to Concentrated Load (ft.)	I-Beam Size
170	10	2000	0	0	?
171	12	?	0	0	10″ 25.4-lb.
172	14	0	28,000	7	?
173	14	0	?	7	9″ 21.8-lb.
174	16	1000	10,000	8	?

34. Relation between Shear and Moment.—In Art. 32 it was shown that

$$V = \frac{dM}{dx},$$

which states that the vertical shear for a section of a beam is equal to

the rate, with respect to the distance along the beam, at which the bending moment is changing at the section. Or, the equation may be written

$$dM = Vdx,$$

which means that the difference, dM (Fig. 116), between the bending moments at two sections that are the distance dx apart is equal to the area, Vdx, under the shear curve between the two sections. Now the difference ($M_2 - M_1$ or ΔM) between the bending moments at the sections x_2 and x_1 would be

$$\Delta M = \int_{M_1}^{M_2} dM$$

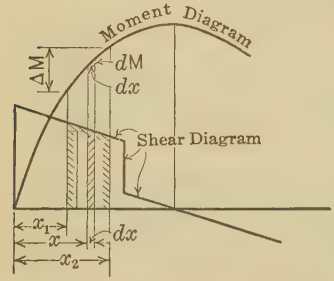


FIG. 116.—Relation between bending moment and vertical shear.

(Fig. 116). But, from the above equation,

$$\int_{M_1}^{M_2} dM = \int_{x_1}^{x_2} Vdx. \quad . \quad (42)$$

And $\int_{x_1}^{x_2} Vdx$ represents the area under the shear curve between the ordinates at the sections x_1 and x_2, and hence,

> $M_2 - M_1$ = area under the shear curve between ordinates at the sections x_1 and x_2. Therefore, the difference between the bending moments at two sections of a beam is represented by the area under the shear curve between ordinates at the two sections.

Thus, the bending moment at a given section of a simple beam is represented by the area under the shear curve between the end of the beam ($M = 0$) and the given section. For example, let it be required to find the maximum moment for a simple beam subjected to a total load W, uniformly distributed over the entire span, l. The shear diagram is shown in Fig. 117. According to the above proposition the difference between the bending moments at the center and end is represented by the area under the shear curve between these two sec-

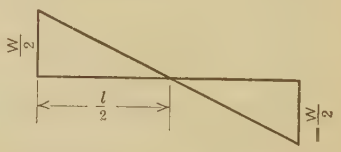


FIG. 117.—Shear diagram.

tions. But the moment at the end is zero and hence the moment at the center is

$$M_c = \frac{1}{2} \text{ base} \times \text{altitude} = \frac{1}{2}\frac{W}{2} \times \frac{l}{2} = \frac{1}{8}Wl,$$

which checks the result found in Prob. 152.

Frequently this proposition may be used conveniently to obtain the moment diagram for a beam from the shear curve, by calculating the areas under the shear curve between as many sections as are necessary to plot satisfactorily the moment curve.

PROBLEMS

175. Show, by the above method, that the maximum bending moment for a simple beam carrying a concentrated load P at mid-span is $M = \frac{1}{4}Pl$.

176. Show, by the above method, that the bending moment at any section within the middle third of a beam when loaded with equal concentrated loads at the third-points is $M = \frac{1}{3}Pl$.

177. Find, by the above method, the maximum moment for the beam shown in Fig. 104.

178. Verify the moment diagram shown in Fig. 98 by the method discussed in the above article.

179. Obtain the moment diagram from the beam shown in Fig. 106 by the method discussed in the above article.

180. Obtain the moment diagram for the beam shown in Fig. 107 by the method discussed in the above article.

181. Obtain the moment diagram for the beam shown in Fig. 115 by the method discussed in the above article.

35. Overhanging Beams.—The discussion in the preceding articles concerning the stresses developed in simple and cantilever beams also applies to overhanging beams. The bending moment at a section in an overhanging beam, however, may be negative, and there are two maximum moments to be considered—the maximum positive moment and the maximum negative moment. In other words, the vertical shear passes through zero at two or more sections of the beam. Further, since the moment changes from a positive to a negative value it must be zero at some section; the section in the beam at which the bending moment is zero is called the *point of inflection.*

The above facts will be discussed in connection with the following illustrative problem.

ILLUSTRATIVE PROBLEM

Problem 182.—A 6-in. by 12-in. by 14-ft. timber beam is supported and loaded as shown in Fig. 118(*a*). (*a*) Draw the shear and moment diagrams. (*b*) Find the position of the dangerous section. (*c*) Find the maximum positive moment. (*d*) Find the maximum negative moment. (*e*) Find the point of inflection. (*f*) Find the maximum fiber unit-stress.

Solution.—The reactions are found to be

$$R_1 = 4600 \text{ lb.} \quad \text{and} \quad R_2 = 7400 \text{ lb.}$$

(*a*) The shear and moment diagrams are shown in Fig. 118(*b*). The vertical shear is equal to zero for a section 3.25 ft. from the left support, and passes through zero at the section above the right support. These two sections, then, are the dangerous sections. The bending moment at any section to the left of the first concentrated load (P_1), at a distance x from the left support, is

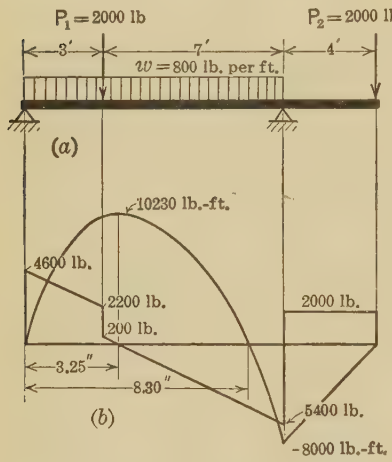


FIG. 118.

$$M_x = 4600x - 800\frac{x^2}{2}, \quad . \quad (43)$$

in which x has any value from 0 to 3. The bending moment at any section between P_1 and the right reaction is

$$M_x = 4600x - 800\frac{x^2}{2} - 2000(x - 3), \quad (44)$$

in which x cannot have a value less than 3 ft. or greater than 10 ft. The bending moment at any section between the right reaction and the right end of the beam is

$$M_x = 4600x + 7400\,(x - 10) - 2000\,(x - 3) - 8000\,(x - 5), \quad . \quad . \quad (45)$$

in which x cannot be less than 10 or greater than 14 ft. Or, if x is measured from the right end of the beam instead of the left end then the bending moment at any section between R_2 and P_2 is

$$M_x = -2000x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Equations (45) and (46) will give, of course, the same value for the bending moment at a given section of the beam.

The moment diagram could also be obtained conveniently from the shear curve by the method discussed in Art 34.

(*b*) The maximum positive moment is

$$M_{pos.} = 4600 \times 3.25 - 2000 \times 0.25 - \frac{800 \times (3.25)^2}{2}$$

$$= 10{,}230 \text{ lb.-ft.}$$

(*c*) The maximum negative moment, obtained from the forces to the left of the section, is

$$M_{neg.} = 4600 \times 10 - 2000 \times 7 - 800 \times 10 \times 5$$

$$= -8000 \text{ lb.-ft.}$$

Or, when obtained from the forces to the right of the section, it is

$$M_{neg.} = -2000 \times 4 = -8000 \text{ lb.-ft.}$$

(*d*) The bending moment changes signs at a section between the dangerous sections. Hence, to locate the inflection point equation (44) may be equated to zero. Thus,

$$M_x = 4600x - 800\frac{x^2}{2} - 2000(x - 3) = 0.$$

Therefore,

$$x = 8.30 \text{ ft.}$$

and hence the inflection point is 8.30 ft. from the left support.

(*e*) Since the maximum positive moment is larger than the maximum negative moment, the maximum fiber unit-stress is

$$s = \frac{Mc}{I} = \frac{10{,}230 \times 12 \times 6}{\frac{1}{12} 6 \times (12)^3} = 852 \text{ lb. per sq. in.}$$

PROBLEMS

For each of the beams described in the following three problems obtain: (*a*) The shear and moment diagrams; (*b*) the maximum positive and negative bending moments; (*c*) the points of inflection; (*d*) the maximum fiber unit-stress in the beam. The weight of the beam may be neglected in each problem.

183. The beam shown in Fig. 119 has the following dimensions: $b = 8$ in., $d = 10$ in., and the loads are, $P = 1000$ lb. and $w = 300$ lb. per ft.

Ans. (*d*) $s = 960$ lb. per sq. in.

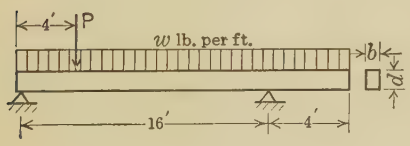


FIG. 119.

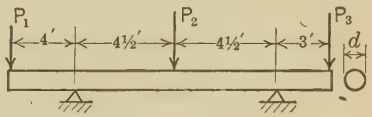


FIG. 120.

184. The diameter, d, of the beam shown in Fig. 120 is 4 in. and the loads are $P_1 = 1500$ lb., $P_2 = 6000$ lb., and $P_3 = 3000$ lb.

185. Refer to the beam shown in Fig. 85 and consider the beam to have a rectangular cross-section 4 in. wide by 12 in. deep.

186. The rolled steel I-beam shown in Fig. 121 rests at A on a soft material which distributes the pressure on the beam uniformly over the surface of contact. The tank and water B weigh 6000 lb. Draw the shear diagram for the beam and calculate the maximum bending moment. Assuming a working fiber unit-stress of 18,000 lb./in.^{2},

calculate the section modulus required and from a steel designer's hand book (or from the tables in Appendix III) select a rolled steel I-beam. *Ans.* $\frac{I}{c} = 36$ in.^{3}

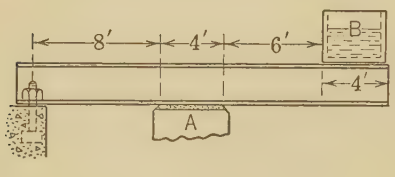


FIG. 121.

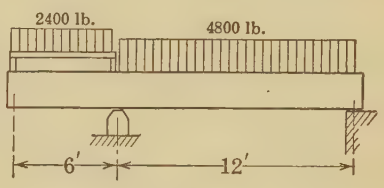


FIG. 122.

187. Draw the shear and moment diagram for the beam shown in Fig. 122. If the beam is made of timber for which the allowable fiber unit-stress is 1000 lb./in.^{2}, what should be the dimensions of the rectangular cross-section assuming that the depth is twice the width. *Ans.* 5.06 in. by 10.12 in.

36. Bending Moments Determined Graphically.—*Culmann's Method.*—The bending moment of the forces acting on a beam about any section may be found by use of the force and string polygons as follows:

In Fig. 123 is shown a beam acted on by four vertical loads P_1 to P_4. By drawing the force and string polygons in the usual way, the reactions R_1 and R_2 are found as indicated in the figure. Now let it be required

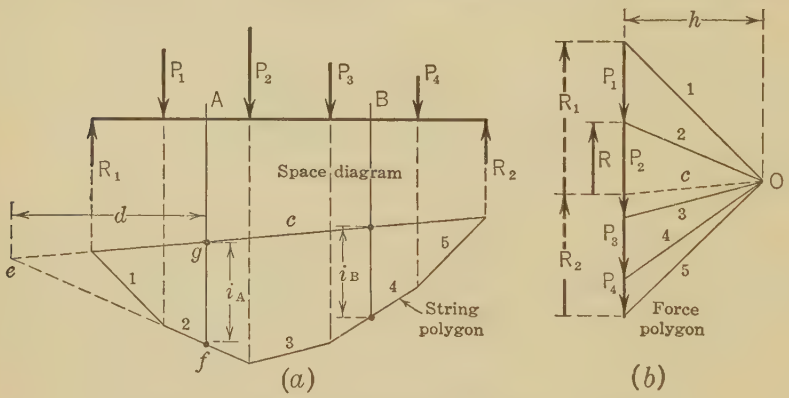


FIG. 123.—Graphical determination of bending moment.

to find the bending moment about the section A. By definition this moment is the algebraic sum of the moments of R_1 and P_1 about A or the moment of the resultant of R_1 and P_1 about A. The magnitude and direction of this resultant is given in the force polygon, denoted by R, and one point on its action line is the intersection of the strings c and 2, denoted by e in the space diagram; the perpendicular distance from A

to this resultant is denoted by d. Therefore the bending moment for the section A is

$$M_A = Rd.$$

Now let a line be drawn through A parallel to the resultant R intersecting the strings c and 2 at g and f. The resulting triangle efg in the string polygon has its sides parallel to the sides of the triangle formed by R, c and 2 in the force polygon. These triangles, therefore, are similar and their altitudes and sides are proportional. Let h denote the perpendicular distance from R to the pole O in the force polygon, and let the intercept gf in the string polygon be denoted by i_A. Then from similar triangles we have

$$\frac{d}{h} = \frac{i_A}{R} \quad \text{or} \quad Rd = hi_A.$$

Therefore,

$$M_A = hi_A.$$

Similarly,

$$M_B = hi_B,$$

in which h is called the pole distance and i the intercept. This relation is not restricted to a parallel system of forces but applies to any coplanar force system. It states that the moment of any set of coplanar forces, about a point A, is equal to the product of the pole distance of the resultant of the forces and the intercept which the strings that represent the two components of the resultant cut from a line drawn through A parallel to the resultant.

Stated in a different way: Ordinates in the bending moment diagram may be found by multiplying the corresponding ordinates (intercepts) in the string polygon by h. Thus, by choosing proper scales (for example, by choosing h equal to unity) the string polygon may be made to represent the bending moment diagram.

It should be remembered that h represents a force and must be measured to the scale of the force polygon, and that i is a length and must be measured to the scale of the space diagram.

For distributed loads (either uniform or varying) this graphical method may be used by dividing the length of the beam over which the load is distributed into short lengths, and assuming the load on each short length to be a concentrated load acting at the center of the length.

37. Economical Sections of Beams.—The efficient use of material in a force-resisting member involves (*a*) the selection of available material that is well suited to the use of the member, and (*b*) the distribution of the material in the member so that it can best resist the forces acting on

the member. The use of material, however, must always be considered in relation to the cost involved.

We are here interested mainly in the effect of the distribution of the material in a beam on the resistance of the beam to external loads; this involves (*a*) the effect of the shape of cross-section and (*b*) the effect of a variation of size of cross-section along the beam.

(*a*) *Effect of Shape of Cross-section.*—If a beam having a constant cross-section safely resists static loads, the maximum fiber unit-stress at the dangerous section must not exceed the allowable or working unit-stress that experiments and experience have shown to be permissible. That is, s in the flexure formula, $s = \frac{Mc}{I}$, must not be greater than the working unit-stress when M is the maximum bending moment.

FIG. 124.—Sections having large values of $\frac{I}{c}$.

The flexure formula shows that, when a given unit-stress s is developed in the beam, the bending moment M required to develop this unit-stress is large when $\frac{I}{c}$ is large. Now $\frac{I}{c}$ is made large by forming the cross-section so that the greater part of the area is as far from the neutral axis as is practicable. Thus, steel beams are rolled in the form of I-sections (Fig. 124*a*), channel-sections (Fig. 124*b*), etc. Built-up steel beams of various shapes (Fig. 124*c* and *d*), and also cast-iron beams are made to conform to this principle. Steel beams are made with the two flanges equal in area since the proportional limits in compression and tension are approximately equal. Cast-iron beams, however, are cast with the tensile flanges larger in area than the compressive flange (see Fig. 110) since the compressive strength of cast iron is much larger (about four times) than the tensile strength. Although cast-iron beams are rarely used in buildings or bridges, they are frequently used in machine frames that are subjected to bending.

(*b*) *Effect of a Varying Section along the Beam.*—As discussed in Art. 33 the bending moment, in general, varies along a beam and is maximum at one section of the beam. If, then, a beam has a constant cross-section $\left(\frac{I}{c} = \text{a constant}\right)$ the maximum fiber unit-stress will occur on the outer

fiber of the section at which the bending moment is maximum, and the unit-stress in the outer fibers at all other sections will be less than that at the dangerous section. Therefore, when the beam is carrying the load that causes the allowable fiber unit-stress in the beam there is much material in the beam on either side of the dangerous section that is under-stressed, and hence this under-stressed material could be saved by varying the cross-section of the beam so that the $\frac{I}{c}$ would vary as the bending moment M varies. This would cause the unit-stresses in the outer fibers of all sections to be equal, since $s = \frac{M}{\frac{I}{c}}$.

Rolled steel beams nearly always are beams of constant cross-section since the cost of rolling a beam of variable cross-section would

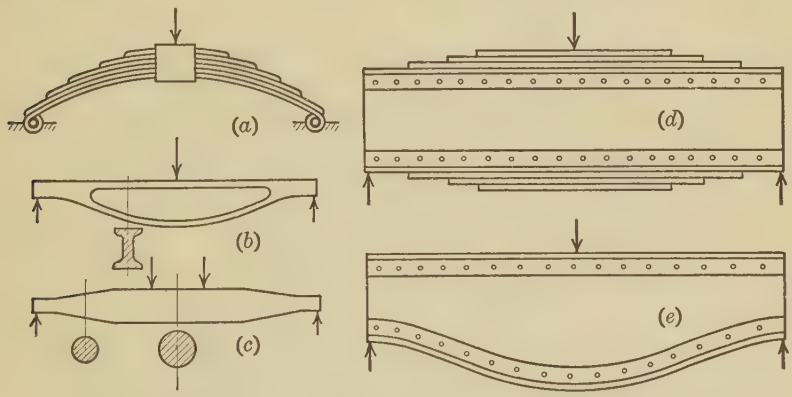


FIG. 125.—Beams that approach the conditions for uniform stress.

offset the saving in material. Built-up beams, such as leaf springs (Fig. 125*a*) and plate girders (Fig. 125*d*), frames of cars (Fig. 125*e*), forged axles (Fig. 125*b*) and turned axles (Fig. 125*c*), etc., are frequently made with a variable section so that the beam approximates a beam of uniform strength.

38. Shearing Stress in a Beam.—In Art. 30 it was shown that in a section of a beam there is shearing stress as well as tensile and compressive stresses, and that the total resisting shearing stress, V_r, in any section is equal to the vertical shear, V, for the section. That is, $V = V_r$. In the preceding articles the tensile (or compressive) unit-stress at any point in the beam was found in terms of the external loads and the dimensions of the beam, and the problem now to be considered

is that of expressing the shearing unit-stress at any point in a beam in terms of the external loads and the dimensions of the beam.

If the shearing unit-stress on the cross-sectional area, a, of a beam were constant and equal to s_s the resisting shear, V_r, would be equal to as_s, and hence the shearing unit-stress at any point of the area would be equal to $\frac{V}{a}\left(s_s = \frac{V}{a}\right)$. But, the shearing unit-stress is not constant over the area, as will be shown later in this article, and hence the equation $s_s = \frac{V}{a}$ gives *only the average* shearing unit-stress. Shearing stress in beams is of importance mainly in timber beams, concrete beams, and some built-up steel beams.

As stated in Art. 27, if a shearing unit-stress occurs on one plane at a point in a body, a shearing unit-stress of equal magnitude must occur at the same point on another plane at right angles to the first plane.

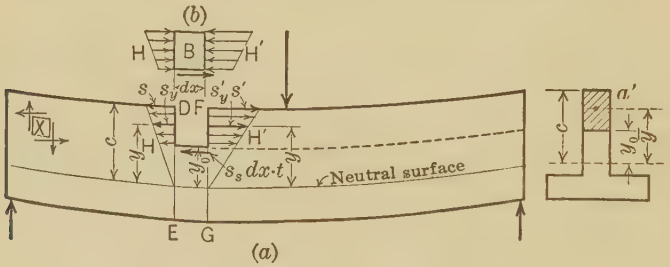


FIG. 126.—Shearing stress in a beam.

Thus, in Fig. 126, X represents a small block in a beam; there is a vertical upward shearing stress on its left face and a vertical downward shearing stress on its right face, but these two forces form a couple which would rotate the block and hence, since the block is in equilibrium, there must be horizontal shearing forces on its upper and lower faces as indicated in Fig. 126. Thus we are led to the conclusion (see Art. 27 for proof) that the vertical and horizontal shearing unit-stresses *at any point* are equal.

The horizontal shearing unit-stress (and also the vertical shearing unit-stress) at any point in a beam may be found, in terms of the external forces and the dimensions of the beam as follows:

Let a block, B, having a small width dx (Fig. 126) be removed from the beam and be replaced by the forces that the block was exerting on the beam. These forces must have been exerted, of course, on the faces that were in contact with the beam. Now, as shown in Art. 31, the compressive (or tensile) unit-stress at any point on any cross-section of

the beam varies directly as the distance of the point from the neutral axis, being zero at the neutral axis. Thus the block B must have pushed horizontally on its two end faces as shown in Fig. 126(a) (or the beam must have pushed on the end faces of the block as shown in Fig. 126b).

But since the bending moments at the two sections are not, in general, equal, the sum of the stresses on the two faces are not equal and hence there must be a shearing stress on the bottom face of the block.

Thus, if H' and H represent the sum of the stresses on the right and left faces, respectively, the total shearing stress on the bottom force is equal to $H' - H$, assuming H' to be larger than H. Further, since the area $dx \cdot t$ on which the shearing stress occurs is small the shearing unit-stress, s_s, may be assumed to be constant over the area and hence the total shearing stress is $s_s \cdot dx \cdot t$. Therefore,

$$s_s dx \cdot t = H' - H.$$

Now $H' = \int_{y_0}^{c} s_y' da$ in which s_y' is the unit-stress on the plane FG on a fiber at the distance y from the neutral axis (Fig. 126a). Further, $\frac{s_y'}{y} = \frac{s'}{c}$ = a constant (Art. 31). Therefore,

$$H' = \frac{s'}{c}\int_{y_0}^{c} y da, \text{ and similarly } H = \frac{s}{c}\int_{y_0}^{c} y da.$$

Hence

$$s_s dx \cdot t = \frac{s' - s}{c}\int_{y_0}^{c} y da.$$

But $s' = \frac{M'c}{I}$ and $s = \frac{Mc}{I}$ in which M' and M are the bending moments at the sections FG and DE, respectively. Therefore,

$$s_s dx \cdot t = \frac{M' - M}{I}\int_{y_0}^{c} y da.$$

But $M' - M = dM$ since the two sections are the small distance dx apart. Hence

$$s_s = \frac{dM}{dx} \cdot \frac{1}{It}\int_{y_0}^{c} y da,$$

and from Art. 32,

$$\frac{dM}{dx} = V.$$

Therefore,

$$s_s = \frac{V}{It}\int_{y_0}^{c} y da, \quad \ldots\ldots\ldots \quad (47)$$

in which s_s is the horizontal shearing unit-stress (and also the vertical shearing unit-stress) in a cross-section for which the vertical shear is V, and at a point whose distance from the neutral axis is y_0, the thickness of the beam at the distance y_0 from the neutral axis being t; I is the moment of inertia of the whole cross-section of the beam about the neutral axis. The expression $\int_{y_0}^{c} yda$ is the first moment (often called the statical moment), about the neutral axis, of that part of the cross-sectional area of the beam between the plane on which the horizontal shearing unit-stress s_s occurs and the outer face of the beam (that is, between y_0 and c). This area is the cross-hatched area a' in the end view in Fig. 126*a*. Further, if the distance, $\bar{y}$, of the centroid of the area a' from the neutral axis is known, the moment of this area may be found from the product $a'\bar{y}$ since

$$\int^{c} yda = a'\bar{y} \quad \text{(Art. 164)}.$$

Hence the above equation may be written

$$s_s = \frac{V}{It} \cdot a'\bar{y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

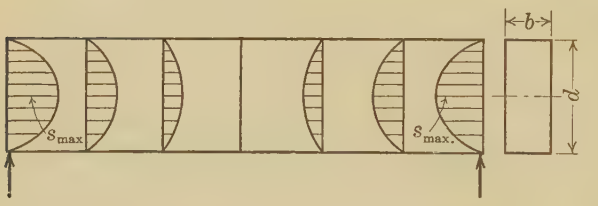


FIG. 127.—Variation of shearing stress in length of beam.

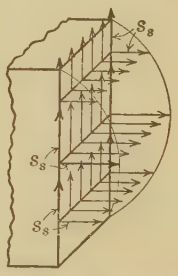


FIG. 128.—Variation of shearing stress over section of beam.

The above equations are valid only when the tensile or compressive unit-stress in the beam does not exceed the proportional limit of the material since the flexure formula, which is based on this assumption, was used in its derivation.

It is important, next, to locate the point in a beam at which the shearing unit-stress, s_s, is a maximum. If the beam has a constant cross-section, I and t are constant, and hence the maximum value of s_s in equations (47) and (48) will occur in the section of the beam for which V is maximum; for a *simple* beam subjected to a uniform load, V is

maximum close to one support. Further, in any section of constant thickness, s_s will be maximum when $\int_{y_0}^{c} yda$ or $a'\bar{y}$ is maximum, which occurs when y_0 is zero, that is, s_s is maximum at the neutral surface. Thus in a *simple* beam of *constant thickness*, subjected to a uniform load the horizontal shearing unit-stress varies throughout the beam as shown in Fig. 127, and at any point in a section the horizontal and vertical shearing unit-stresses are equal as indicated in Fig. 128.

Thus, the maximum shearing unit-stress in any section of a beam having a *rectangular* cross-section bd (Fig. 127) is

$$s_s = \frac{V}{It}a'\bar{y} = \frac{V}{\frac{1}{12}bd^3 \cdot b} \cdot b\frac{d}{2} \times \frac{d}{4}$$

$$= \frac{3}{2}\frac{V}{bd} = \frac{3}{2}\frac{V}{a},$$

in which a is the area of the whole cross-section. Hence the maximum value s_s in any section of a *rectangular* beam is 50 per cent greater than the average shearing unit-stress in the section. If a beam has a *circular* cross-section the maximum value of s_s is $33\frac{1}{3}$ per cent greater than $\frac{V}{a}$, that is, $s_s = \frac{4}{3}\frac{V}{a}$.

ILLUSTRATIVE PROBLEMS

Problem 188.—Find the maximum shearing unit-stress in a simple beam composed of two 15-in. 45-lb. channels placed back to back and used on a 12-ft. span. The beam is subjected to a concentrated load of 50,000 lb. at the center of the span. (Neglect the weight of the beam.) The dimensions of each channel are shown in Fig. 129.

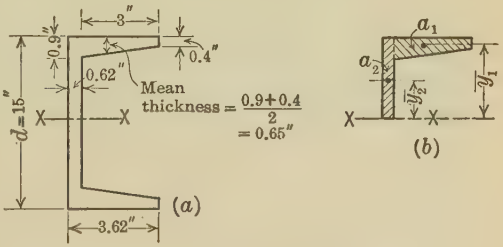


FIG. 129.—15-in. 45-lb. channel section.

Solution.—A steel maker's handbook gives the following values in addition to those shown in Fig. 129(*a*).

$I_x = 375$ in.^{4} $a = 13.2$ sq. in.

The maximum shearing unit-stress is $s_s = \frac{V}{It}a'\bar{y}$

$V = 25{,}000$ lb. $I = 750$ in.^{4} $t = 1.24$ in.

The value of $a'\bar{y}$ is most easily found as the sum of the moments of the areas a_1 and a_2 (Fig. 129*b*) for both channel sections. Thus,

$$a'\bar{y} = 2a_1\bar{y}_1 + 2a_2\bar{y}_2$$
$$= 2(3 \times 0.65) \times \left(7.5 - \frac{0.65}{2}\right) + 2(0.62 \times 7.5) \times 3.75 = 63.0 \text{ in.}^3$$

Therefore

$$\text{Max. } s_s = \frac{25{,}000}{750} \cdot \frac{63.0}{1.24} = 1693 \text{ lb./in.}^2$$

The maximum bending unit-stress $\left(s = \frac{Mc}{I}\right)$ is found to be 18,000 lb./in.^{2} Thus the maximum shearing unit-stress is small in this section (and in nearly all other rolled steel sections except when used on very short spans) even though the bending (normal) stress is the full allowable value.

The average shearing unit-stress is

$$s_{\text{av.}} = \frac{V}{a} = \frac{25{,}000}{26.4} = 947 \text{ lb./in.}^2$$

and hence the maximum shearing unit-stress is 1.79 times the average.

Shearing Stress in Web of I-beam.—It is customary in structural design to assume that the flanges of a channel-beam or an I-beam are effective in resisting bending but not in resisting shear, and the maximum shearing unit-stress in the beam is assumed to be equal to the average shearing unit-stress over the web only, the web being regarded as extending the entire depth of the beam. Thus in the above problem

$$(s_s)_{\text{av.}} \text{ in web} = \frac{V}{td} = \frac{25{,}000}{1.24 \times 15} = 1344 \text{ lb. per sq. in.}$$

The justification for assuming the shearing unit-stress in an I-beam to be distributed uniformly over the web considered to extend the entire depth of the beam may be found in a study of Fig. 130. In any section of an I-beam s_s, according to equation (47), will be large where t is small, and small where t is large, the effect on s_s of the term $\int_{y_0}^{c} y\,da$ being relatively small in the case of an I-section; further, s_s will change abruptly where t changes abruptly. Hence, the horizontal (and vertical) shearing unit-stress at a section in an I-beam varies

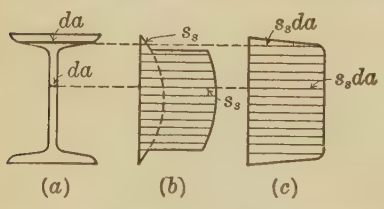


FIG. 130.—Distribution of shearing stress over section of I-beam.

approximately as shown in Fig. 130(*b*), and the total shearing stress ($s_s \cdot da$) on the elementary strips of the section (the total stress per unit of depth) is approximately constant for the whole depth of the section as shown in Fig. 130(*c*).

PROBLEMS

189. A simple beam made of wood has a cross-section 6 in. wide by 8 in. deep. A concentrated load of 16,000 lb. acts at the mid-span section, and the span is 10 ft. (*a*) Find the maximum shearing unit-stress. (*b*) Find also the shearing unit-stress at a point 2 in. from the top of the beam in a section 2 ft. from the left support.

Ans. (*a*) $s_s = 250$ lb. per sq. in. (*b*) $s_s = 188$ lb. per sq. in.

190. A simple timber beam has a hollow square cross-section; the outside dimensions are 8 in. by 8 in. and the inside dimensions are 4 in. by 4 in. The span of the beam is 10 ft. and the beam is subjected to a uniformly distributed load of 1500 lb. per ft. Find the maximum shearing unit-stress. Where in the beam does this stress occur?

Ans. $s_s = 328$ lb. per sq. in.

191. A Douglas fir beam (Fig. 131) having a cross-section 8 in. by 16 in. when tested as a simple beam with a span of 15 ft. by equal concentrated loads at the third-points, failed by shear (as shown in Fig. 131) when the total load on the beam was 48,850 lb. Find the maximum shearing unit-stress in the beam when failure occurred.

Ans. $s_s = 286$ lb. per sq. in.

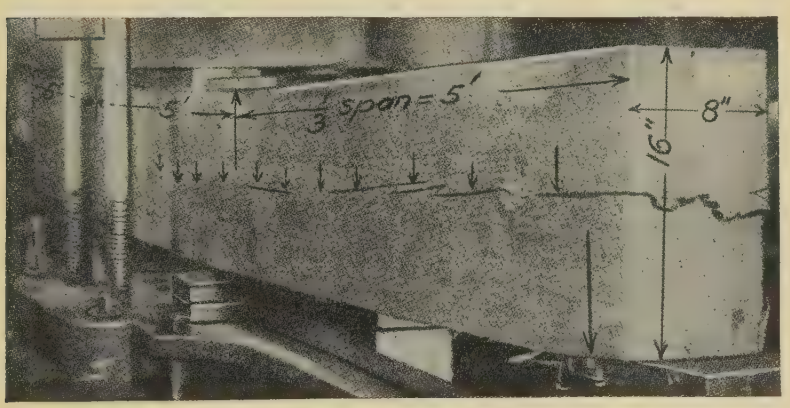


FIG. 131.—Timber beam that failed by horizontal shear.

192. Calculate the maximum shearing unit-stress in the 9-in. *I*-beam shown in Fig. 365 if used as a simple beam on a span of 12 ft. and loaded with a uniformly distributed load which causes a maximum tensile unit-stress of 16,000 lb. per sq. in.

Ans. $s_s = 3690$ lb. per sq. in.

193. A large Douglas-fir beam is to be designed to resist a concentrated load of 20,000 lb. at the center of the span. Use a working stress equal to one-third of the ultimate shearing unit-stress found in Problem 191 (286 lb. per sq. in.). If the span of the beam is 12 ft. and the width of the beam is to be 8 in. what should be the depth

of the beam? A working shearing stress of 100 lb. per sq. in. is frequently specified for large Douglas-fir beams, and 125 lb. per sq. in. for dense pine.

194. Compare the flexural strengths of the two beams illustrated in Fig. 132; the beams have rectangular cross-sections. In (*a*) the beam is composed of two equal parts of the same material; one part slides over the other because no shearing resistance is developed on the surfaces of contact, and hence the bending resistance of the beam is the sum of the resistances of the two beams acting independently. In (*b*)

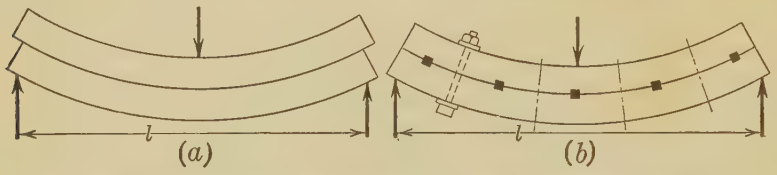


FIG. 132.—Effect of shearing resistance in a beam composed of two parts.

shearing resistance is exerted by the keys and bolts, and hence the two parts of the beam act as one beam.

195. A timber beam with a span of 4 ft. is subjected to a concentrated load of 12,800 lb. at the center of the span. The beam has a cross-section of 8 in. by 12 in. It is specified that the tensile unit-stress in the beam shall not exceed 1200 lb./in.^{2} and that the shearing unit-stress shall not exceed 100 lb./in.^{2} Will the specifications be satisfied if (*a*) the 8-in. side is placed vertically? (*b*) If the 12-in. side is placed vertically? (*c*) If the span were 20 ft. and the load at the center were 3840 lb., which side should be placed vertically?

39. Stress Beyond Proportional Limit. Modulus of Rupture.—As shown in Art. 34 the value of s in the flexure formula, $M = \frac{sI}{c}$, is the tensile or compressive unit-stress in the beam *only when s does not exceed the proportional limit of the material.* If the bending moment M causes a unit-stress greater than the proportional limit of the material the bending moment is still held in equilibrium by the resisting moment, but the resisting moment is not given by the expression $\frac{sI}{c}$ since this expression was found by assuming that the unit-stress at any point varies directly as the distance of the point from the neutral axis; and this is true only when the maximum unit-stress does not exceed the proportional limit.

The distribution of stress on the cross-section of a beam when the maximum unit-stress on the section is less and greater than the proportional limit are shown in Figs. 133(*a*) and (*b*), respectively, and may be explained as follows: Experiments have shown that a plane section of a beam remains approximately plane after the beam is stressed even though the unit-stress exceeds the proportional limit of the material,

and hence the unit-strain at any point is proportional to the distance of the point from the neutral axis in both beams of Fig. 133. In the beam that is stressed beyond the proportional limit (Fig. 133*b*) the unit-stress is not proportional to the unit-strain except near the neutral axis where the strain is small, and hence the unit-stress does not vary directly as the distance from the neutral axis except for short distances from the neutral axis; when the unit-stress on any fiber exceeds the proportional limit the unit-stress on that fiber increases at a smaller rate than does the strain; the curve showing the distribution of unit-stress on the section is the same as a portion of the stress-strain diagram for the material.

Modulus of Rupture.—From Fig. 133 it is evident that with a given unit-stress in the outer fiber of the beam the resisting moment is larger when the unit-stress varies as shown in Fig. 133(*b*) than it would be if the unit-stress varied directly as the distance from the neutral axis. Therefore, the resisting moment in a beam when stressed to a given unit-

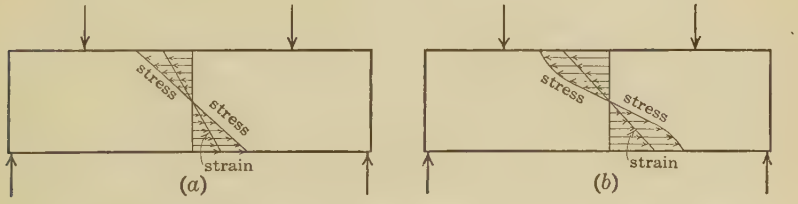


FIG. 133.—Change in distribution of stress when proportional limit is exceeded.

stress beyond the proportional limit is larger than that found from the expression $\frac{sI}{c}$ by substituting the given unit-stress for s. Thus, if the ultimate strength of the material is substituted for s in the flexure formula $M = \frac{sI}{c}$, the resulting value of M will be less than the maximum moment that the beam can resist.

Tests of beams of various material have shown that the maximum bending moment a beam can resist may be from 20 to 100 per cent greater, depending on the material and the shape of cross-sections, than the value of M found from the flexure formula by substituting the ultimate strength of the material for s.

If, then, the allowable or working unit-stress in a beam is taken to be some proportion ($\frac{1}{5}$, say) of the tensile or compressive ultimate strength (using the lesser value) the maximum bending moment that could be applied to the beam would be greater than five times the moment that would produce the working stress. For some materials

(particularly brittle materials) a working unit-stress based on the ultimate strength of the material seems unnecessarily small and this fact is sometimes employed to justify larger working stresses in beams than is used for tension members. The value of s found from the flexure formula by substi uting for M the value of the maximum bending moment that a beam resists when tested to rupture is called the *modulus of rupture* of the material in flexure. Thus, if it is denoted by s_r, we have

$$s_r = \frac{M_{\max.} c}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

It should be noted that s_r is not the unit-stress in the material caused by the bending moment $M_{\max.}$; nor is it the ultimate strength of the material.

But although the modulus of rupture is not the actual maximum unit-stress in the beam, the relative values of the actual unit-stresses in two beams when stressed beyond the proportional limit may be found from the moduli of rupture provided that the two beams have cross-sections of the same shape; and only slight error will occur if the beams have cross-sections of different shapes provided that the cross-sections are symmetrical with respect to the neutral axis and that the stress-strain curves of the material in tension and compression have the same slopes (moduli of elasticity in tension and compression are equal). If these two conditions are not satisfied the neutral axis shifts when the beam is stressed beyond the proportional limit and does not pass through the centroid of the section.

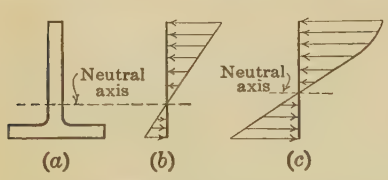


FIG. 134.—Shifting of neutral axis in unsymmetrical section when proportional limit is exceeded.

For example, in a beam with an unsymmetrical section (Fig. 134a), the most remote fibers on one (compressive) side of the neutral axis will be stressed to their proportional limit before those on the other side, as suggested by Fig. 134(b), and after the proportional limit has been exceeded, in order to keep the sum of the stresses on the fibers above the neutral axis equal to the sum of the stresses below the neutral axis, the neutral axis must shift toward the tension side, Fig. 134(c), thereby changing the values of c and I.

Further, if the beam has a symmetrical section but is made of material, such as cast iron, whose fibers deform more in tension than in compression when subjected to relatively large unit-stresses, the

neutral axis must shift toward the compressive side of the beam in order to keep the sum of the tensile stresses equal to the sum of the compressive stresses, and this shifting of the neutral axis increases the resistance of the beam.

40. Maximum Moment Due to Moving Loads.—Let several concentrated loads move or roll over a beam so that the distance between the loads remain constant, as for example, the wheel loads of a locomotive when moving on a bridge, and let it be required to find the position of the loads when they cause the greatest bending moment on the beam.

When the loads are in any given position on the beam the dangerous section must occur under one of the loads since the vertical shear will pass through zero under one of the loads, but although the bending moment is maximum at a section beneath a certain one of the loads (W_2, say, Fig. 135) when the loads are in a given position, the bending moment at a section beneath the same load (W_2) after the loads have been moved to another position may be greater or less than it was when the loads were in the previous position.

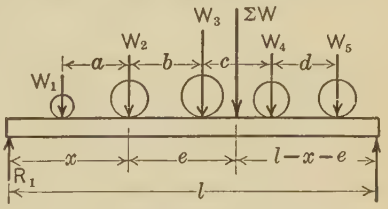


FIG. 135.—Moving loads.

In Fig. 135 let W_1, W_2, W_3, W_4, and W_5 be concentrated loads that remain fixed distances apart as they roll over the beam of span l, and let it be required to find the position of W_2 such that the bending moment at a section beneath W_2 will be greater than for any other position of W_2.

The resultant of the loads is ΣW and it will be assumed to act at the distance e from W_2. Now from one of the equations of equilibrium we have,

$$R_1 = \frac{l - x - e}{l}\Sigma W.$$

Let x denote the distance of W_2 from the left support, then the bending moment at the section beneath W_2 is

$$M_x = \frac{l - x - e}{l}x\Sigma W - W_1 a.$$

Now the value of x which will make M_x a maximum may be found by equating to zero the first derivative of M_x with respect to x. Thus,

$$\frac{dM_x}{dx} = \frac{l - 2x - e}{l}\Sigma W = 0.$$

Hence

$$\frac{l - 2x - e}{l} = 0,$$

or

$$x = \frac{l}{2} - \frac{e}{2}, \quad \ldots \ldots \ldots \quad (50)$$

and hence W_2 *will be in its position of maximum moment when it is as far to one side of the center of the span as the resultant of all the loads on the span is to the other side of the center of the span.* And the same statement will apply to each of the loads.

In order to find the greatest bending moment to which the beam is subjected, each load in turn must be put in its position of maximum bending moment and the bending moment at the section beneath the load must be found; the greatest of these maximum moments is the *greatest* bending moment to which the beam is subjected.

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196. Two loads, 4000 lb. and 2000 lb., 6 ft. apart roll over a simple beam 12 ft. long. Find the position of the loads to give the maximum bending moment and design a yellow-pine beam to carry this load, using a working stress of 1200 lb. per sq. in. Investigate for shear. *Ans.* 4 in. by 14 in.

197. Three loads roll over a simple beam having a span of 20 ft. The distance between the first (front) load and the second load is 8 ft., and between the second and third is 4 ft. The first load is 8000 lb. and each of the others is 4000 lb. Using a working stress of 16,000 lb./in.^{2}, calculate the section modulus for the beam.

41. Assumptions and Limitations Involved in the Flexure Formula. —In the derivation of the flexure formula, $M = \frac{sI}{c}$, several assumptions were made which impose limitations on the use of the formula. These assumptions and limitations may be summarized as follows:

1. The unit-stress on any fiber of the beam is proportional to the unit-strain of the fiber, and hence the maximum fiber unit-stress in the beam does not exceed the proportional limit. This assumption of proportionality of stress and strain is a close approximation to the law of behavior of most engineering materials except cast iron and concrete. And, for these materials the assumption does not as a rule introduce serious error in the flexure formula when the unit-stress in the beam is not greater than the usual working stress.

2. The beam is composed of material for which the modulus of elasticity in tension is the same as that in compression. This assumption

is a reasonably close approximation to the results of experiments for most engineering materials, except cast iron. And the error involved in the case of cast iron, although considerable, is not as a rule serious at working stresses. The flexure formula does not, of course, apply directly to a beam made of two or more different materials such as a reinforced concrete beam, but can be made to apply to such a beam by using an equivalent or transformed homogeneous section (see Chap. XX).

3. The axial strain (stretch or shortening) of any fiber in the beam is proportional to the distance of the fiber from the neutral surface. This involves the further assumption that the effect of shearing strain on the axial strain is negligible; which, except for very short deep beams, introduces little error, particularly since the maximum longitudinal fiber stress occurs at the section on which the shearing stress is zero.

4. The *unit*-strain as well as the total strain of any fiber of the beam is proportional to the distance of the fiber from the neutral axis. This assumption requires that all fibers shall have the same length before bending, that is, the beam shall be straight, and hence the flexure formula does not apply to curved beams. (See Chap. XVI.)

5. The loads act in one plane which contains the centroidal axis of the beam; the loads are perpendicular to the centroidal axis of the beam; and the neutral surface is perpendicular to the plane of the loads. These assumptions require that the plane of the loads shall contain an axis of symmetry of each cross-section, in which case the neutral axis of any section is the other axis of symmetry (or the other principal axis of inertia of the cross-section). The flexure formula does not apply, therefore, to a beam loaded unsymmetrically (see Chap. XV).

6. The proportions of the beam are such that the beam fails (elastically) by bending and not by twisting, lateral collapse, local wrinkling, etc. For example, a rectangular beam $\frac{1}{4}$ in. wide by 12 in. deep would probably fail by twisting, and an I-beam having very wide and thin flanges would probably fail by local wrinkling of the flange, etc.

Most of the steel, timber and other one-material beams that occur in structures and machines conform approximately to the conditions on which the flexure formula is based. Further, the flexure formula is applied frequently to beams (such as curved beams, unsymmetrically loaded beams, flat plates, etc.), that do not satisfy the conditions required to make its application valid; this practice is often justified if the amount of the error is known approximately, and if allowance is made for the error in some way as, for example, by reducing the working stress, or by introducing a correction factor in the formula.

REVIEW QUESTIONS AND PROBLEMS

198. Five of the following six statements are wrong. Correct the statements.

(*a*) A statically indeterminate beam is one to which the equations of equilibrium do not apply.

(*b*) The bending moment for any section of a beam is the algebraic sum of the moments, about a line in the section, of all the forces acting on the beam.

(*c*) The dangerous section of a beam is the section for which the vertical shear is a maximum.

(*d*) The equation $V dx = dM$ states that the difference in bending moments at two sections of a beam the distance dx apart is represented by the area under the vertical shear curve between the ordinates erected at the two sections.

(*e*) The unit-stress on a fiber of a beam is proportional to the unit-strain of the fiber because each plane (transverse) section of the beam before the beam is bent is a plane section after bending.

(*f*) The maximum horizontal (and vertical) shearing unit-stress in a beam of constant cross-section occurs at the neutral axis on a section for which the bending moment is maximum.

199. Give the facts and reasoning leading to the conclusion that the longitudinal unit-strains of the fibers of a beam at any section are proportional to the distances of the fibers from the neutral axis of the section.

200. Assuming that the normal unit-stress at any point on a transverse section of a beam is proportional to the distance of the point from the neutral axis of the section, show that the neutral axis passes through the centroid of the section.

201. A beam having a rectangular cross-section 4 in. wide by 12 in. deep has the same section modulus as another beam whose cross-section is circular. What is the diameter of the circular cross-section? *Ans.* $d = 9.92$ in.

202. A simple beam having a span of 12 ft. is subjected to a single concentrated load of P lb. 4 ft. from the left support. The cross-section is a channel section 12 in. wide and 8 in. deep, thickness of the flange and the web being 2 in. The beam is placed with the web on the bottom and the flanges extending upward. Calculate the allowable value of P if the allowable value of the normal (tensile and compressive) bending unit-stresses are 4000 lb. per sq. in. and 8000 lb. per sq. in., respectively, and the allowable value of the shearing unit-stress is 1000 lb./in.^{2}

Ans. $P = 11{,}300$ lb. for normal stress; $P = 32{,}600$ lb. for shear.

203. A timber beam having a rectangular cross-section 6 in. wide by 12 in. deep is used as a simple beam on a span of 8 ft. It supports two equal concentrated loads, placed so that there is a load 2 ft. from each end (loaded at the quarter points). Calculate the maximum allowable value of the loads assuming values of the allowable unit-stresses to be 1000 lb. per sq. in. for the normal stress and 100 lb. per sq. in. for the shearing stress. *Ans.* Each load = 4800 lb.

204. Draw the shear and moment diagrams for the beam shown in Fig. 136, neglecting the weight of the beam. If the beam is made of steel and the section modulus is 18 in.^{3}, what is the maximum bending unit-stress in the beam?

Ans. $s = 16{,}700$ lb. per sq. in.

205. Each of the two steel beams shown in Fig. 137 is $\frac{1}{2}$ in. deep and 4 in. wide. AB is a frictionless guide surface along which the block C slides when the load P is

applied. Each beam is fixed at both ends. What load P will produce a flexural unit-stress of 16,000 lb. per sq. in.? (Hint: Consider each beam to be equivalent to two cantilever beams; that is, assume that each beam is sawed in two at the point of inflection and that each part is acted on by the resisting shearing force at this section.) *Ans.* $P = 222$ lb.

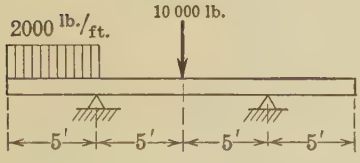

FIG. 136.—Overhanging beam.

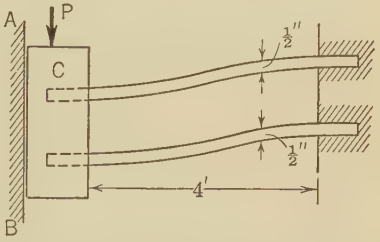

FIG. 137.—A beam fixed at both ends equivalent to two cantilevers.

206. Given that the bending moment about section AA (Fig. 138) is $-$ 15,400 lb. ft. and that the vertical shear for this section is 6700 lb. Calculate the bending moment and vertical shear for section BB. *Ans.* $+$ 10,400 lb.-ft.

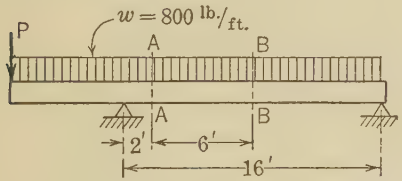

FIG. 138.—Bending moments at any two sections of a beam.

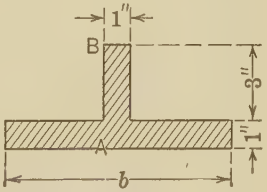

FIG. 139.—Tee section.

207. The cross-section of a beam is a T-section as shown in Fig. 139. On any section the flexural unit-stress at A is one-third of that at B, all stresses in the beam being less than the proportional limit of the material. What must be the width, b, of the flange? *Ans.* $b = 9$ in.

CHAPTER V

DEFLECTION OF BEAMS

(Double Integration Method [1])

42. Introduction.—Beams in structures and machines when resisting static loads must have stiffness as well as strength; that is, beams must resist the loads without permitting too great an elastic deflection. In fact stiffness is sometimes the governing factor in the design of the beam.

Now the deflection of a beam depends on (1) the loads acting on the beam, (2) the stiffness of the material of which the beam is made, and (3) the dimensions of the beam. The stiffness of a material is measured by the modulus of elasticity of the material (Art. 6).

The purpose of this chapter is to determine in what way and to what extent the elastic deflection of a statically determinate beam depends on the loads, the stiffness of the material, and the dimensions of the beam, when the beam is loaded in various ways. To do this the general equation of the elastic curve of a beam will first be found.

43. Elastic Curve Equation.—The elastic curve of a beam is the curve of the longitudinal centroidal axis of a stressed beam, provided that the maximum unit-stress in the beam does not exceed the proportional limit of the material; since the elastic curve lies in the neutral surface of the beam it does not change its length as the beam is bent.

The general equation of the elastic curve of a beam may be derived as follows: Fig. 140 represents a beam [2] bent by loads that cause a maximum fiber stress less than the proportional limit of the material (the deflection of the beam is exaggerated in the figure). As discussed in Art. 31, when a straight beam is bent the strains of the fibers are proportional to the distances of the fibers from the neutral surface. Thus in Fig. 140 GH and $E'F'$ are any two sections that were parallel before

[1] See Chapter XII for a discussion of the moment-area method of determining deflections of beams.

[2] It is assumed that the beam is straight before the loads are applied and that the plane of the loads contains an axis of symmetry of each cross-section of the beam so that the neutral plane is perpendicular to the plane of loads (see Art. 41). Further, the beam is assumed to be subjected to pure bending; that is, the deflection resulting from shearing deformation is assumed to be negligible.

the beam was bent; the distance $\overline{AB}$ between the two parallel sections is assumed to be a differential length and will be denoted by dl, where l is the length of the elastic curve. By drawing EF through B parallel to GH (and hence parallel to the original position of $E'F'$) it is evident that the upper fiber $\overline{HF}$ has shortened the amount $\overline{FF'}$ and the bottom fiber has elongated the amount $\overline{EE'}$, denoted by de in Fig. 121, and the strains of the other fibers are proportional to the distances of the fibers

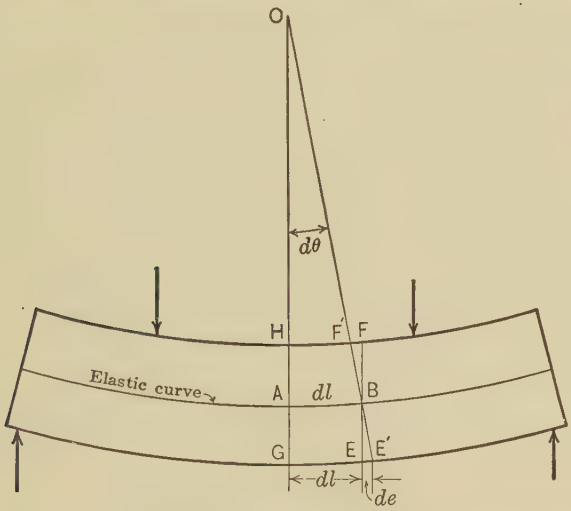

FIG. 140.—Deflection of a beam.

from the neutral surface. Further, the unit-strain, ϵ, of the bottom fiber is

$$\epsilon = \frac{de}{dl}.$$

Now, the sectors OAB and BEE' are similar, and hence

$$\frac{\overline{BE}}{\overline{OA}} = \frac{\overline{EE'}}{\overline{AB}}.$$

But $\overline{OA}$ is the radius of curvature of the elastic curve at the point A, denoted by ρ, and $\overline{BE}$ is the distance from the neutral surface to the fiber whose strain is de, denoted by c as in Art. 34. Hence

$$\frac{c}{\rho} = \frac{de}{dl} = \epsilon; \quad \rho = \frac{c}{\epsilon}. \qquad (51)$$

But c and ϵ are related to the external forces, the stiffness of the

material, and the dimensions of the beam according to the following equations:

$$M = \frac{sI}{c} \text{ (Art. 31)} \quad \text{and} \quad E = \frac{s}{\epsilon} \text{ (Art. 6).}$$

Therefore, equation (51) may be written:

$$\rho = \frac{EI}{M} \quad \text{or} \quad M = \frac{EI}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

in which ρ is the radius of curvature of the elastic curve at a section for which the bending moment is M; E is the modulus of elasticity of the material, and I is the moment of inertia of the cross-section of the beam about the neutral axis; if E is expressed in pounds per square inch, I in inches^4 and ρ in inches, M will be expressed in pound-inches.

It should be noted from the above equation that if a beam of constant cross-section is so loaded that the bending moment M is constant over a portion of the beam, the radius of curvature of the elastic curve of this portion will also be constant (since E and I are constant), and hence the elastic curve for this portion is an arc of a circle. Conversely, if a beam is bent in an arc of a circle the bending moments for all sections of the beam are equal. Further, the above equation also shows that when M is equal to zero, ρ is equal to infinity; thus, at the inflection point ($M = 0$, Art. 35) the center of curvature is at an infinite distance from the beam.

PROBLEMS

208. A band saw 1/20 in. thick by 1 in. wide is in contact with the circumference of a wheel having a diameter of 50 in. Find the bending moment to which the saw is subjected and the unit-stress in the outer fibers. *Ans.* $s = 30{,}000$ lb. per sq. in.

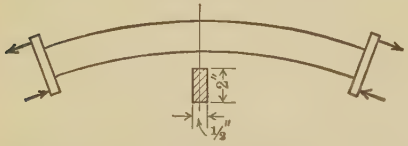

FIG. 141.—Beam subjected to couples.

FIG. 142.—Beam composed of two bars.

209. A steel rod is subjected to bending couples at its ends as shown in Fig. 141. If the moment of each couple is 2400 lb. ft. and the cross-section of the rod is $\frac{1}{2}$ in. by 2 in., with the 2-in. dimension in the plane of the couple, find the radius of curvature of the rod. The proportional limit is 90,000 lb. per in.^{2}

210. A simple beam (Fig. 142) is composed of two rectangular bars of the same width and made of the same material. The depth of the top bar is $\frac{1}{2}$ in. and that of the bottom bar is 1 in. The beam is subjected to load causing no stress greater than the proportional limit of the material. Compare the maximum bending unit-stresses in the two bars. Hint: the radii of curvature of the two beams at any section are equal.

211. Find the radius of curvature at the section AB of the beam described in Problem 139, assuming the beam to be made of yellow pine (consult Table 1).

Ans. 6550 in.

Elastic Curve Equation Expressed in Rectangular Coordinates.—The expression for the radius of curvature, ρ, is (see any textbook on calculus)

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Now for beams that are straight before being bent and that are not deflected more than is usual in structural and machine members, the value of the slope $\frac{dy}{dx}$ is always small[3] compared to unity, and hence $\left(\frac{dy}{dx}\right)^2$ may be neglected without introducing serious error. Thus, the above expression for ρ becomes

$$\rho = \frac{1}{\frac{d^2y}{dx^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Therefore, the equation $M = \frac{EI}{\rho}$ may be written

$$M = EI\frac{d^2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

which is the general equation of the elastic curve of a beam. In this equation M is the bending moment at the section whose distance from the origin of the coordinates is x, and y is the deflection of the elastic curve at the same section. The sign to be selected for the right-hand member of the equation is discussed below.

In order to determine the value of the deflection y for any given value of x, M is expressed in terms of x and the differential equation is then integrated twice; the equation thus found, of course, will depend on the type of beam (simple, cantilever, etc.) and the type of loading (concentrated, uniformly distributed, etc.). The equations of the elas-

[3] For example, the tangent to the elastic curve would probably never be as large as one part in 20 and hence the error in equation (53) would not exceed about four parts in 1000.

tic curves and the maximum deflections of beams of various types are found in Art. 44 to 48.

Signs of M and $\frac{d^2y}{dx^2}$.—In using the elastic curve equation, $M = EI\frac{d^2y}{dx^2}$, in this and the following chapters, it is important to understand the significance of the signs of M and $\frac{d^2y}{dx^2}$; E and I are essentially positive and may be regarded merely as magnitudes. Now the sign of M has already been discussed (Art. 33); it is positive for a horizontal beam when it produces tensile stress in the bottom fibers of the beam, or when it causes the center of curvature to lie above the beam, and is negative when it causes compressive stress on the bottom fibers, etc. The sign

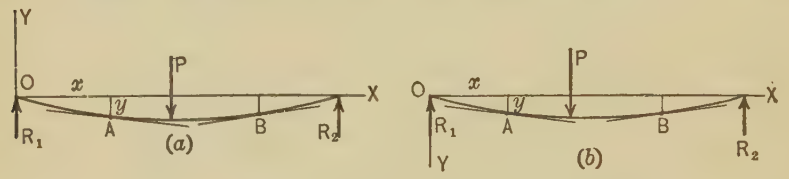

FIG. 143.—Effect of direction of axes on sign of $\frac{d^2y}{dx^2}$.

of $\frac{d^2y}{dx^2}$, however, depends on the choice of the positive direction of the axis. For example, Fig. 143*a* represents a horizontal simple beam subjected to a positive bending moment. Let the origin of axes be chosen at the left end of the beam and let the positive directions for the x and y axes be to the right and upwards, respectively.

Now the slope $\frac{dy}{dx}$ at a point A on the curve is negative whereas at a point B the slope is positive, and thus as x increases $\frac{dy}{dx}$ increases. Therefore, the rate of change of $\frac{dy}{dx}$ with respect to x $\left(\text{that is, } \frac{d^2y}{dx^2}\right)$ is positive, and since M is also positive, the equation may be written

$$M = EI\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

If, however, the positive direction of the y-axis were chosen downward as shown in Fig. 143(*b*), the slope at A would be positive and would decrease as x increases; thus $\frac{d^2y}{dx^2}$ would be negative. But M is positive

and the right side of the equation must, then, also be positive, which requires that the equation shall be written

$$M = -EI\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

If, then, for any beam the x-axis is positive to the right and the positive direction of the y-axis is chosen opposite to the direction of the deflection, equation (56) should be used since the sign of $\frac{d^2y}{dx^2}$ will be the same as that of M (for a cantilever both signs will be negative), and if the positive direction of the y-axis is chosen the same as that of the deflection, equation (57) should be used, since the sign of $\frac{d^2y}{dx^2}$ will be opposite to that of M.

Principle of Superposition.—In this and some of the following chapters use will be made of the principle of superposition which states that the effect on an elastic body caused by any one of several loads (whether the effect is stress, moment, deflection, etc.) does not influence the effect caused by another load, and hence the resultant effect can be found from the effect caused by each load as if acting alone. The principle is applicable to most members provided that only small (elastic) strains are involved.

44. Deflection of Simple Beam, Uniform Load.—Let it be required to determine the equation of the elastic curve and the maximum deflection of a simple beam, having a span l, when subjected to a uniformly

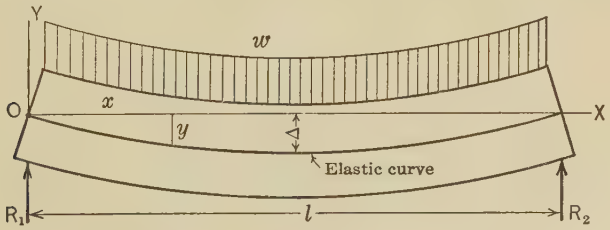

FIG. 144.—Deflection of simple beam subjected to uniform load.

distributed load of w per unit length (Fig. 144); the known quantities in addition to w and l are the modulus of elasticity E and the moment of inertia, I, of the cross-section of the beam.

Let the y-axis be chosen positive upward as shown in Fig. 144. The general elastic curve equation, then, is $EI\frac{d^2y}{dx^2} = M$. But M varies

with x and may be expressed in terms of x; thus, the bending moment at any section whose distance from the left support is x, is

$$M_x = R_1 x - wx \cdot \frac{x}{2} = \frac{wl}{2}x - \frac{wx^2}{2}.$$

Substituting this expression for M in the general elastic curve equation we have

$$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

By integrating we obtain,

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + c_1,$$

in which c_1 is a constant of integration. The value of a constant in an equation may be determined by substituting a pair of values of the variables; in this equation the variables are the slope $\left(\frac{dy}{dx}\right)$ of the elastic curve and the distance x. From inspection, when

$$x = \frac{l}{2}, \quad \frac{dy}{dx} = 0.$$

By substituting this pair of values in the above equation the value of c_1 is found. Thus

$$c_1 = -\frac{wl^3}{16} + \frac{wl^3}{48} = -\frac{1}{24}wl^3.$$

The above equation, then, becomes

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{1}{24}wl^3. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

By integrating this equation and evaluating the constant of integration, the equation expressing the relation between y and x is found. Thus,

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + c_2.$$

From inspection, when $x = 0$, $y = 0$, and hence $c_2 = 0$.

Therefore, the equation of the elastic curve of a simple beam subjected to a uniformly distributed load is

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Maximum Deflection.—Now the maximum deflection, Δ, occurs at the mid-span, that is, y in the above equation becomes Δ when x is equal to $l/2$. Hence the maximum deflection is

$$\Delta = \frac{1}{EI}\left(\frac{wl^4}{96} - \frac{wl^4}{384} - \frac{wl^4}{48}\right).$$

$$\Delta = -\frac{5}{384}\frac{wl^4}{EI} = -\frac{5}{384}\frac{Wl^3}{EI}. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The minus sign shows that the deflection is opposite to the positive direction of the y-axis. If the total load W is expressed in pounds, l in inches, E in pounds per square inch, and I in inches^4, Δ will be expressed in inches.

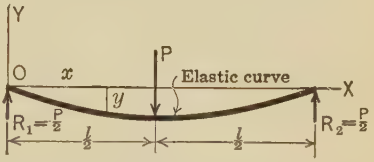

FIG. 145.—Deflection of simple beam subjected to concentrated load at mid-span.

45. Deflection of Simple Beam, Concentrated Load at Mid-span.—Let the axes be chosen as shown in Fig. 145. For any section in the left half of the beam,

$$M = R_1 x = \tfrac{1}{2}Px.$$

Hence

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}Px. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

By integrating we obtain

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + c_1;$$

when $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$, and hence $c_1 = -\frac{Pl^2}{16}$.

Thus,

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{Pl^2}{16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

By integrating again we obtain

$$EIy = \frac{Px^3}{12} - \frac{Pl^2}{16}x + c_2; \quad . \quad . \quad . \quad . \quad . \quad (64)$$

when $x = 0$, $y = 0$, and hence[4] $c_2 = 0$.

[4] It should be noted that although the right support does not deflect, the value of y in equation (64) is not zero when x is equal to l, since $l/2$ is the greatest value x can have in the equation; for values of x greater than $l/2$ the expression $\frac{1}{2}Px$ cannot be used for M in eq. (62).

Therefore, the equation of the elastic curve of the left half of the beam is

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Since the maximum deflection, Δ, occurs at the mid-span, that is, when $x = \frac{l}{2}$, the maximum deflection is

$$\Delta = \frac{1}{EI}\left(\frac{Pl^3}{96} - \frac{Pl^3}{32}\right) = -\frac{1}{48}\frac{Pl^3}{EI}. \quad . \quad . \quad . \quad . \quad (66)$$

The minus sign shows that the deflection is opposite to the positive direction of the Y-axis.

Equation (65) is also the equation of the elastic curve of the right half of the beam if the origin of axes is taken at the right end of the beam and the positive direction of x is to the left.

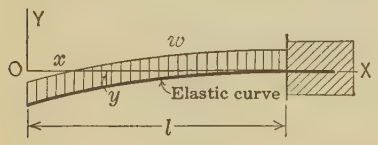

FIG. 146.—Deflection of cantilever beam subjected to uniform load.

46. Deflection of Cantilever Beam, Uniform Load.—Let the uniformly distributed load be w per unit length and let the axes be chosen as in Fig. 146.

Since M is negative the elastic curve of the beam is

$$EI\frac{d^2y}{dx^2} = M = -\frac{wx^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

By integrating and determining constants as in the preceding articles we have

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + c_1;$$

when $x = l$, $\frac{dy}{dx} = 0$; therefore, $c_1 = \frac{wl^3}{6}$.

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

$$EIy = -\frac{wx^4}{24} + \frac{wl^3}{6}x + c_2;$$

when $x = l$, $y = 0$; therefore, $c_2 = -\frac{1}{8}wl^4$

The elastic curve equation then is

$$EIy = -\frac{wx^4}{24} + \frac{wl^3}{6}x - \frac{1}{8}wl^4. \quad . \quad . \quad . \quad . \quad (69)$$

Now the maximum value of y in the above equation occurs when x is equal to 0. Hence the maximum deflection is

$$\Delta = -\frac{1}{8}\frac{wl^4}{EI} = -\frac{1}{8}\frac{Wl^3}{EI}. \quad . \quad . \quad . \quad . \quad . \quad (70)$$

PROBLEMS

212. Show that the magnitude of the maximum deflection of a cantilever beam when subjected to a concentrated load P at its free end is

$$\Delta = \frac{1}{3}\frac{Pl^3}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

213. A 15-in. 42.9-lb. I-beam is used as a cantilever beam. The length of the cantilever is 10 ft. and the beam is subjected to a uniformly distributed load. Find the deflection of the free end of the beam when the maximum fiber unit-stress in the beam is 16,000 lb. per sq. in.; neglect the weight of the beam. *Ans.* $\Delta = 0.256$ in.

214. A piece of pine having a cross-section 2 in. wide by 4 in. deep is used as a cantilever beam to resist a concentrated load P at the free end. The beam has a free length of 4 ft. It is required that the maximum deflection (see Problem 212) shall not exceed 0.5 in. and that the maximum fiber unit-stress shall not exceed 1000 lb. per sq. in. What is the maximum value of P? Neglect the weight of the beam. The modulus of elasticity of the pine may be assumed to be 1,500,000 lb. per sq. in.

215. A simple beam subjected to a uniformly distributed load or to a concentrated load at mid-span may be considered to be two cantilever beams, each cantilever being fixed at the mid-span section of the simple beam. Let the beam in Fig. 145 be considered in this way and by use of eq. (71) show that eq. (66) is correct for the maximum deflection of the beam in Fig. 145.

216. Find the deflection, in terms of w, l, E, and I, of a point on the elastic curve of the beam in Fig. 144, which is at the distance $l/4$ from the left support.

217. Find the slope to the elastic curve of the beam in Fig. 145 at a point whose distance from the left support is $l/4$; express the slope in terms of P, l, E, and I.

Ans. $-\frac{3}{64}\frac{Pl^2}{EI}$.

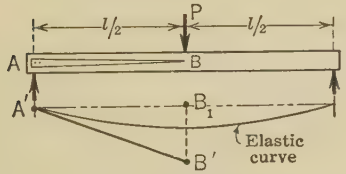

FIG. 147.

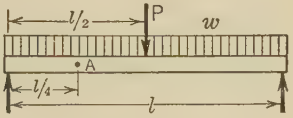

FIG. 148.

218. A light pointer AB (Fig. 147) is attached at A to a simple beam by two nails a short distance apart. Find the total deflection B_1B' of the end of the pointer in terms of P, l, E and I. *Ans.* $-\frac{Pl^3}{32EI}$.

219. Find the slope to the elastic curve at the point A of the beam shown in Fig. 148. The known quantities are P, w, l, E and I.

220. Calculate the maximum deflection of the beam shown in Fig. 148, assuming that the beam is made of steel and that the cross-section is rectangular, 2 in. wide and 6 in. deep. Assume also that $P = 3000$ lb., $w = 60$ lb./ft. and $l = 10$ ft.

221. A simple beam 12 ft. long is subjected to a uniformly distributed load of 400 lb. per ft. The cross-section of the beam is 4 in. wide by 8 in. deep. The beam is made of oak having a modulus of elasticity of 2,000,000 lb. per sq. in. and a proportional limit of 3500 lb. per sq. in. It is specified that the maximum fiber unit-stress shall not exceed 2000 lb. per sq. in. and that the maximum deflection shall not exceed $\frac{1}{360}$ of the span. Are the requirements satisfied?

222. How much will a 12-in. 55-lb. I-beam deflect when stressed to its proportional limit of 40,000 lb. per sq. in. by a concentrated load at the mid-span if the span is 20 ft.? *Ans.* $\Delta = 1.07$ in.

223. A cylindrical steel shaft 4 in. in diameter is used as a simple beam on a span of 12 ft. The beam is subjected to a uniformly distributed load of 200 lb. per ft. and a concentrated load of P lb. at the center of the span. If the maximum deflection of the beam must not exceed 0.40 in. and the maximum fiber unit-stress must not exceed 20,000 lb. per sq. in., find the maximum value of P.

Ans. $P = 2290$ lb. for strength; $P = 940$ lb. for stiffness.

224. In Art. 45 use for M the bending moment for any section to the *right* of the load and show that the magnitude of the maximum deflection is $\frac{1}{48}\frac{Pl^3}{EI}$. Use the same origin as in Fig. 145.

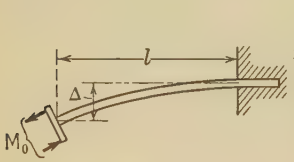

FIG. 149.

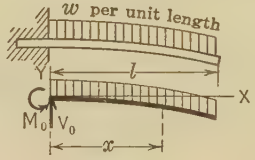

FIG. 150.

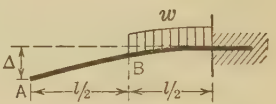

FIG. 151.

225. A beam (Fig. 149) is fixed at one end and is subjected to a bending couple M_0 at the free end. Find the deflection of the free end in terms of M_0, l, E, and I. Neglect the weight of the beam.

226. Find the deflection of the free end of the cantilever beam shown in Fig. 150, by choosing the origin at the fixed end and the axes as indicated in the figure. The known quantities are w, l, E and I.

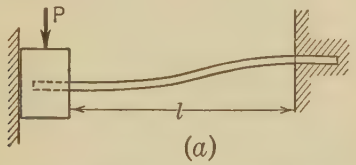

FIG. 152.—Deflection of cantilever beam subjected to a concentrated load and a couple at the free end.

227. Find the slope of the mid-point B of the beam shown in Fig. 151, and the deflection of the free end A in terms of w, l, E and I.

Ans. $\theta_B = \frac{wl^3}{48EI}$; $\Delta = 0.0182\frac{wl^4}{EI}$.

228. Determine the elastic curve equation of the beam shown in Fig. 152(*a*),

which is a beam subjected to a bending couple and a concentrated load at each end; the beam is fixed at both ends. The known quantities are P, M_0, E and I.

47. Deflection of Overhanging Beam, Uniform Load.—Fig. 153(a) represents a beam that overhangs the same amount at each end and is subjected to a uniformly distributed load of w per unit length over the entire length of the beam. Let it be required to find the equation of the elastic curve of the central portion of the beam, and the maximum deflection of the beam.

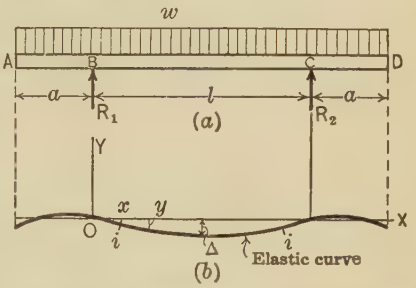

FIG. 153.—Deflection of overhanging beam.

Let the axes be chosen as shown in Fig. 153(b). The reaction R_1 is found by applying one of the equations of equilibrium as follows:

$$\Sigma M_c = 0 = R_1 l + \frac{wa^2}{2} - \frac{w(a+l)^2}{2}.$$

$$R_1 = \tfrac{1}{2}wl + wa.$$

The bending moment at any section between B and C (Fig. 153a) is

$$M_x = R_1 x - \frac{w(a+x)^2}{2}$$

$$= -\tfrac{1}{2}wx^2 + \tfrac{1}{2}wlx - \tfrac{1}{2}wa^2. \quad . \quad . \quad . \quad . \quad . \quad (72)$$

For all points between B and C:

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wlx - \frac{1}{2}wa^2. \quad . \quad . \quad . \quad . \quad (73)$$

By integrating this equation twice and making use of the following conditions,

$$\frac{dy}{dx} = 0 \text{ when } x = \frac{l}{2};\ y = 0 \text{ when } x = 0;\ y = y_{\max.} = \Delta \text{ when } x = \frac{l}{2},$$

the elastic curve equation of the central portion of the beam is found to be

$$EIy = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wlx^3 - \tfrac{1}{4}wa^2x^2 - \tfrac{1}{24}wl^3x + \tfrac{1}{4}wa^2lx; \quad (74)$$

and the maximum deflection is found to be

$$\Delta = -\frac{5}{384}\frac{wl^4}{EI} + \frac{1}{16}\frac{wl^2a^2}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

48. Deflection of Simple Beam, Concentrated Load Not at Mid-span. —If a concentrated load P acts on a beam the expressions for M on opposite sides of the load are different whether the load acts at the mid-span or not, and hence there are two elastic curves which have different equations. But, if the load does not act at the mid-span the constants of integration cannot be determined as in Art. 45; they are found by making use of the fact that the two elastic curves have a common tangent and a common ordinate under the load. Further, the maximum deflection of the beam is the maximum value of y in the equation of only one of the curves.

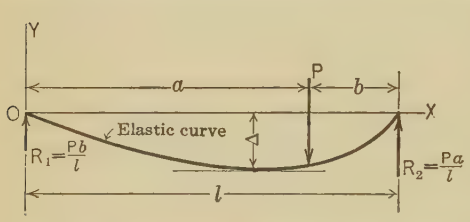

FIG. 154.—Deflection of simple beam; concentrated load at any point.

Fig. 154 represents a simple beam subjected to a concentrated load P at the distance a from the left support and b from the right support, a being greater than b. Let it be required to find the equation of the elastic curve of the left portion of the beam and the maximum deflection of the beam. Let the axes be chosen as shown in Fig. 154.

The bending moment at any section to the left of P is

$$M = R_1 x = \frac{Pb}{l}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

in which x may have any value from 0 to a. And the moment at any section to the right of P is

$$M = \frac{Pb}{l}x - P(x - a), \quad . \quad . \quad . \quad . \quad . \quad (77)$$

in which x cannot be less than a nor greater than l.

For all points to the left of the load:

$$EI\frac{d^2y}{dx^2} = \frac{Pbx}{l}. \quad . \quad . \quad . \quad (78)$$

$$EI\frac{dy}{dx} = \frac{Pbx^2}{2l} + c_1. \quad . \quad (79)$$

For all points to the right of the load:

$$EI\frac{d^2y}{dx^2} = \frac{Pb}{l}x - P(x - a). \quad . \quad . \quad (80)$$

$$EI\frac{dy}{dx} = \frac{Pb}{2l}x^2 - \frac{P(x-a)^2}{2} + c_3. \quad (81)$$

And since the curves have a common tangent under the load, when x

is made equal to a in both (79) and (81) the value of $\frac{dy}{dx}$ in (79) is equal to $\frac{dy}{dx}$ in (81). Thus

$$\frac{Pba^2}{2l} + c_1 = \frac{Pba^2}{2l} - \frac{P(a-a)^2}{2} + c_3;$$

whence,

$$c_1 = c_3.$$

Now by substituting c_1 for c_3 in (81) and by integrating (79) and (81) there is found:

$$EIy = \frac{Pbx^3}{6l} + c_1 x + c_2; \quad \ldots\ldots\ldots \quad (82)$$

when $x = 0$, $y = 0$, hence, $c_2 = 0$.

$$EIy = \frac{Pbx^3}{6l} - \frac{P(x-a)^3}{6} + c_1 x + c_4. \quad \ldots \quad (83)$$

Since the curves have a common ordinate under the load, when $x = a$ in (82) and (83) the values of y in these equations are equal, and hence

$$\frac{Pba^3}{6l} + c_1 a = \frac{Pba^3}{6l} + c_1 a + c_4.$$

$$c_4 = 0.$$

Now when $x = l$ in (83), $y = 0$, and hence

$$c_1 = -\frac{Pbl^2}{6l} + \frac{P(l-a)^3}{6l} = -\frac{Pb}{6l}(l^2 - b^2). \quad \ldots \quad (84)$$

By substituting this value of c_1 in (82) the equation of the elastic curve of the left portion of the beam is found to be

$$EIy = \frac{Pbx^3}{6l} - \frac{Pb(l^2 - b^2)x}{6l}. \quad \ldots\ldots \quad (85)$$

The value of x that makes $\frac{dy}{dx}$ equal to zero is the value of x that makes y in (85) a maximum, and this value of y is the value of the maximum deflection. But the value of $\frac{dy}{dx}$ is given by (79) and hence equating (79) to zero we have

$$x^2 = \frac{l^2 - b^2}{3} = \frac{a(a+2b)}{3}.$$

By substituting this value of x in (85) the maximum deflection is found to be

$$\Delta = -\frac{Pb(l^2 - b^2)\sqrt{3(l^2 - b^2)}}{27EIl}, \quad . \quad . \quad . \quad . \quad . \quad (86)$$

$$\Delta = -\frac{Pba(a + 2b)\sqrt{3a(a + 2b)}}{27EIl}. \quad . \quad . \quad . \quad . \quad (87)$$

If in the above expression a and b are each made equal to $\frac{l}{2}$, that is, if P acts at the mid-span the value of Δ is $\frac{1}{48}\frac{Pl^3}{EI}$ as was found in Art. 45.

PROBLEMS

229. Derive the equations of the elastic curves of a simple beam of length l when loaded with two equal concentrated loads, PP, each load being at a distance of $\frac{1}{4}l$ from a support. Also find the maximum deflection of the beam. Select the origin of axes at the left end of the beam and the positive directions of the x and y axes to the right and upwards, respectively.

$$Ans. \begin{cases} EIy = \dfrac{Px^3}{6} - \dfrac{3}{32}Pl^2x \text{ for the left portion,} \\ EIy = \dfrac{Plx^2}{8} - \dfrac{Pl^2x}{8} + \dfrac{Pl^3}{384} \text{ for the central portion,} \\ \Delta = -\dfrac{11}{384}\dfrac{Pl^3}{EI}. \end{cases}$$

HINT.—The elastic curves of the left and central portions of the beam have a common tangent and a common ordinate under the left load.

230. In the above problem let each load act at the distance $\frac{1}{3}l$ from a support, and find the maximum deflection of the beam.

231. A 10-in. 40-lb. steel I-beam is used as a simple beam on a span, l, of 16 ft. A concentrated load P is applied at a point $\frac{1}{4}l$ from the left support. If the maximum bending unit-stress developed is 16,000 lb./in.^{2}, what is the maximum deflection of the beam? *Ans.* 0.305 in.

232. A timber beam having a rectangular cross-section 6 in. wide by 8 in. deep is used as a simple beam on a span of 15 feet. A load P applied at a section 5 ft. from the left support causes a maximum flexural unit-stress of 1000 lb./in.^{2} Calculate the deflection at the section beneath the load. Assume $E = 10^6$ lb./in.^{2}

233. A 9-in. 30-lb. I-beam rests on two supports 14 ft. apart and resists a concentrated load of 15,000 lb. at a distance of 5 ft. from the left support. Determine whether the following requirements are satisfied.

(*a*) Fiber unit-stress not to exceed 16,000 lb. per sq. in.
(*b*) Maximum deflection not to exceed $\frac{1}{360}$ of the span.

49. General Expression for Maximum Deflection.—It will be well at this point to summarize some of the results obtained in this and the preceding chapter and to note the general expression for the maximum bending moment and maximum deflection for a beam. In the following table, W is the total load on the beam having a span l, E is the modulus of elasticity of the material, and I is the moment of inertia of the cross-section of the beam about the centroidal axis, the cross-section being assumed constant.

Type of Beam and of Loading	Maximum Bending Moment M	Maximum Deflection Δ
Cantilever, Load at end	$1Wl$	$\frac{1}{3}\frac{Wl^3}{EI}$
Cantilever, Uniform load	$\frac{1}{2}Wl$	$\frac{1}{8}\frac{Wl^3}{EI}$
Simple beam, Load at center	$\frac{1}{4}Wl$	$\frac{1}{48}\frac{Wl^3}{EI}$
Simple beam, Uniform load,	$\frac{1}{8}Wl$	$\frac{5}{384}\frac{Wl^3}{EI}$
Any beam, Any load	αWl	$\beta\frac{Wl^3}{EI}$

in which α and β are constants depending on the type of beam and of loading.

If W is eliminated from the general expressions, $M = \alpha Wl$ and $\Delta = \beta\frac{Wl^3}{EI}$, we have $\Delta = \frac{\beta}{\alpha}\frac{Ml^2}{EI}$. But $M = \frac{sI}{c}$ and hence

$$\Delta = \frac{\beta}{\alpha}\frac{s}{E} \times \frac{l^2}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

One way of interpreting this equation is as follows: If several beams are of the same type, are made of the same material, and are subjected to the same maximum bending unit-stress (in other words, if $\frac{\beta}{\alpha}\frac{s}{E} = a$ constant), then the deflections of the beams will vary directly with the squares of the spans and inversely with the distances from the neutral axis to the most remote fibers, in which the maximum unit-stresses occur.

50. Deflection of Beam Due to Shear.—The deflection of a beam due to the shearing stresses in the beam was assumed, in the preceding discussions, to be negligible. In short deep beams, however, the deflection due to shear may require consideration. An approximate value for the deflection due to the shearing stresses may be found as follows:

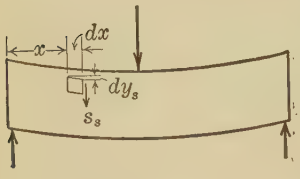

FIG. 155.—Deflection of beam due to shear.

The shearing deformation dy_s, of a small block, of length dx, in the beam (Fig. 155) is

$$dy_s = \epsilon_s dx = \frac{s_s}{E_s} dx,$$

in which s_s is the shearing unit-stress on the face of the block; the deflection of a fiber of length x (Fig. 155) is

$$y_s = \frac{1}{E_s} \int_0^x s_s dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

If s_s is constant throughout the length of all the fibers (as in Fig. 155) the deflection of the beam at a section the distance x from the end is

$$y_s = \frac{s_s x}{E_s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

In general, however, s_s varies over the section, but if an average value of s_s (see Art. 38) is used in equation (90), an approximate value of the deflection due to shear may be found which may be helpful in estimating to what extent the shearing stresses contribute to the deflection of a beam.

A more exact analysis may be made by substituting for s_s in equation (89) the value given by equation (47) of Art. 38.

Since the average shearing unit-stress $s_s = V/a$, equation (90) may be written

$$y_s = \frac{Vx}{aE_s} = \frac{M}{aE_s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

where M is the bending moment at the section at the distance x from the end of the beam.

REVIEW QUESTIONS AND PROBLEMS

234. Three of the following five statements are wrong. Point out and correct the errors.

(*a*) The equations $M = \frac{EI}{\rho}$ and $EI \frac{d^2y}{dx^2} = M$ are equations of the elastic curve of a beam.

(*b*) The expression $\frac{d^2y}{dx^2}$ in the above equation is the radius of curvature of the beam at any section the distance x from the origin.

(*c*) In deriving the equations in (*a*) use was made of the assumption that a plane section of the beam before bending remains a plane section after bending.

(*d*) The M in the equations in (*a*) is the maximum moment in the beam.

(*e*) The maximum elastic deflection of a beam always occurs at the section for which the bending moment is a maximum.

235. The maximum deflection of a beam, in general, may be expressed as $\Delta = k\frac{Wl^3}{EI}$ where k is a constant depending on the type of beam and of loading.

(*a*) What is the meaning of W? (*b*) For a simple beam subjected to a concentrated load at midspan, show that this expression may be written $\Delta = \frac{1}{12}\frac{Ml^2}{EI}$ where M is the maximum bending moment for the beam.

236. Calculate the maximum deflection of a steel shaft 4 in. in diameter used as a simple beam on a span of 20 ft. to support a uniformly distributed load that causes a maximum bending stress of 16,000 lb. per sq. in. *Ans.* 1.60 in.

237. A simple beam carrying a uniformly distributed load may be considered to be composed of two cantilever beams (see Fig. 156), each cantilever having a fixed end at the midspan section of the simple beam and being acted on by an upward concentrated load R_1 at the free end and a uniformly distributed downward load as indicated in Fig. 156(*b*). Starting with the general equation of the elastic curve, and choosing axes as indicated in Fig. 156(*b*), show that the maximum deflection Δ_1 of each cantilever beam is numerically the same as that of the simple beam

$$\left(\Delta_1 = \Delta = \frac{5}{384}\frac{Wl^3}{EI}\right).$$

Check the result by using the principle of superposition and the values in the table in Art. 49.

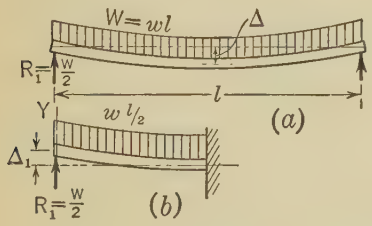

FIG. 156.

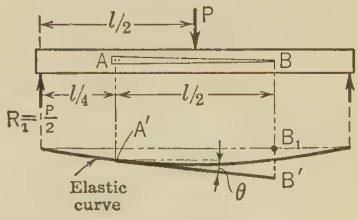

FIG. 157.

238. A light pointer AB (Fig. 157) is attached at A, by two nails a short distance apart, to a simple beam that supports a concentrated load P at the mid-span section. Find the slope (tan θ) of the pointer $A'B'$ and the deflection B_1B' of the end B of the pointer in terms of P, l, E, and I.

Ans. $\tan\theta = \frac{3}{64}\frac{Pl^2}{EI}$, $B_1B' = \frac{29}{768}\frac{Pl^3}{EI}$.

CHAPTER VI

STATICALLY INDETERMINATE BEAMS

(Double Integration Method) [1]

51. Introduction.—In the preceding two chapters the beams considered were held in equilibrium by external forces that formed a system of parallel forces in a plane, and the reactions of the supports were found by applying the *two* equations of equilibrium for such a force system; since there were not more than two supports and hence not more than two unknown external forces acting on the beam, the *two* equations of equilibrium were sufficient to determine all the unknown external forces. In other words, the force systems acting on the simple, cantilever, and overhanging beams considered in the preceding chapters were *statically determinate.*

On the other hand, although fixed-ended beams and continuous beams are held in equilibrium by parallel forces in a plane, yet the reactions of the supports cannot be found from the *two* equations of equilibrium *alone,* since the number of unknown reactions is greater than two. Such beams are said to be *statically indeterminate,* and in finding the reactions use is made of the relation between the external forces and the internal strains produced by those forces (as expressed by the elastic curve equation) in addition to the relations between the external forces alone (as expressed by the equations of equilibrium).

Method of Procedure.—The method of obtaining equations that involve the unknown reactions, in addition to the two equations of equilibrium, is as follows:

In the elastic curve equation, $M = \pm EI \frac{d^2y}{dx^2}$, the expression for M will contain some of the unknown reactions (forces and couples) acting on the beam, and after the elastic curve equation is integrated, a constant of integration being added with each integration, the unknown reactions and constants of integration are determined by substituting in the equations as many pairs of values of the variables as there are unknown reactions and constants of integration; the pairs of values of the variables

[1] See Chapter XIII for a discussion of the moment-area method.

are found from the physical conditions that the beam satisfies. The remaining unknown forces (that did not occur in the expression for M) may then, as a rule, be found from the two equations of equilibrium.

Thus, the equation of the elastic curve may be used for two different purposes: (1) To find the deflection of a beam when all the forces acting on the beam are known, as discussed in Chapter V, and (2) to determine reactions of supports when they cannot be found from the equations of equilibrium alone, as is discussed in this chapter in connection with fixed-ended and continuous beams. The use of the elastic curve equation for the latter purpose is of great importance, for the reactions must be found before the unit-stress at any point, and the deflection at any section can be determined.

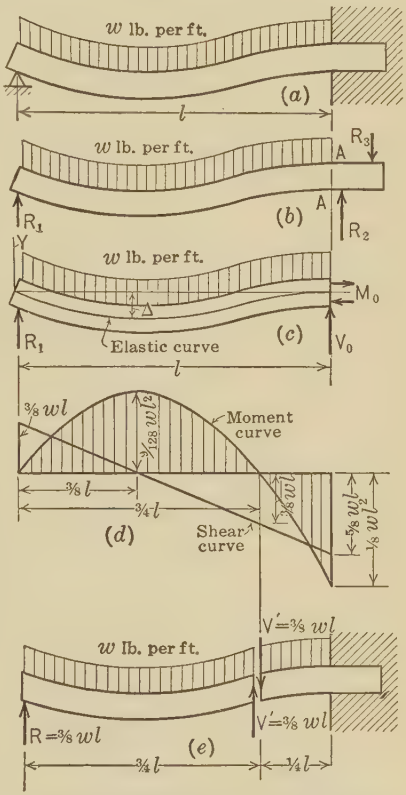

FIG. 158.—Beam fixed at one end, supported at other end; load uniformly distributed.

52. Beam Fixed at One End, Supported at Other End; Uniform Load.—Fig. 158(a) represents such a beam, the tangent line to the elastic curve at the wall being horizontal; the length of span is l, and the load per unit of length is w. As shown in Fig. 158(b), the beam is held in equilibrium by parallel forces of which all three reactions, R_1, R_2, and R_3, are unknown. Thus it is necessary, as explained in the preceding article, to use the elastic curve equation in addition to the two equations of equilibrium in order to determine the reactions.

Now the forces R_2 and R_3 are equivalent to an upward force and a bending couple, and since we are concerned mainly with the part of the beam outside the wall, it is convenient to assume that the beam is cut off even with the face of the wall at the section AA (Fig. 158b) and that R_2 and R_3 are replaced by the forces (stresses) which they cause on this section; namely, the upward shearing force V_0 and the bending couple M_0 as shown in Fig. 158(c). It should be observed that V_0 and M_0 are

the resisting shear and resisting moment, respectively, at the section and are equal in magnitude but opposite in sense to the vertical (external) shear and bending (external) moment for the section.

The three reactions to be found, then, are R_1, V_0 and M_0. The value of R_1 may be found from the elastic curve equation, and V_0 and M_0 may then be found from the two equations of equilibrium as follows:

If axes are chosen as shown in Fig. 158(*c*) the elastic curve equation (see Art. 43) is

$$EI\frac{d^2y}{dx^2} = M = R_1x - \frac{wx^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Integrating, we have

$$EI\frac{dy}{dx} = \frac{R_1x^2}{2} - \frac{wx^3}{6} + c_1;$$

when $x = l$, $\frac{dy}{dx} = 0$ and hence,

$$c_1 = -\frac{R_1l^2}{2} + \frac{wl^3}{6}.$$

Therefore,

$$EI\frac{dy}{dx} = \frac{R_1x^2}{2} - \frac{wx^3}{6} - \frac{R_1l^2}{2} + \frac{wl^3}{6}. \quad . \quad . \quad . \quad . \quad . \quad (93)$$

Integrating again, we obtain

$$EIy = \frac{R_1x^3}{6} - \frac{wx^4}{24} - \frac{R_1l^2}{2}x + \frac{wl^3}{6}x + c_2; \quad . \quad . \quad . \quad (94)$$

when $x = 0$, $y = 0$ and hence, $c_2 = 0$:
when $x = l$, $y = 0$ and hence,

$$0 = \frac{R_1l^3}{6} - \frac{wl^4}{24} - \frac{R_1l^3}{2} + \frac{wl^4}{6}.$$

Therefore,

$$R_1 = \tfrac{3}{8}wl.$$

Now from the two equations of equilibrium ($\Sigma F = 0$ and $\Sigma M = 0$), V_0 and M_0 may be found. Thus,

$$\Sigma F = \tfrac{3}{8}wl + V_0 - wl = 0; \qquad \text{whence, } V_0 = \tfrac{5}{8}wl.$$

$$\Sigma M_A = \tfrac{3}{8}wl \cdot l - wl \cdot \frac{l}{2} + M_0 = 0; \text{ whence, } M_0 = \tfrac{1}{8}wl^2.$$

But, as noted above, the vertical shear and bending moment at the wall, however, are equal *but opposite* to the reactions V_0 and M_0, and hence are negative, the vertical shear being downward and the bending

moment causing tensile stress on the top side of the beam. Thus if V_0 and M_0 denote the vertical shear and bending moment at the wall, we have (as indicated in Fig. 158*d*)

$$V_0 = -\tfrac{5}{8}wl \quad \text{and} \quad M_0 = -\tfrac{1}{8}wl^2.$$

The reader should also obtain these values by simply writing the expressions for the vertical shear and bending moment for the section at the wall in terms of the forces to the left of the section.

Shear and Moment Diagrams. Maximum Positive Moment.—The vertical shear for a section at distance x from the left support is

$$V_x = \tfrac{3}{8}wl - wx,$$

and this equation shows that the shear is zero when $x = \tfrac{3}{8}l$; the shear diagram is shown in Fig. 158(*d*).

The bending moment for a section at the distance x from the left support is

$$M_x = \tfrac{3}{8}wlx - wx\cdot\frac{x}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

The value of the bending moment is a maximum at the section for which the vertical shear is zero (Art. 32), and hence the maximum positive bending moment is

$$\text{Pos. } M_{\max.} = \tfrac{3}{8}wl\cdot\tfrac{3}{8}l - w\cdot\tfrac{3}{8}l\cdot\tfrac{1}{2}\,\tfrac{3}{8}l$$

$$= \tfrac{9}{128}wl^2.$$

The inflexion point is found by equating M_x to zero. Thus,

$$\tfrac{3}{8}wlx - \frac{wx^2}{2} = 0; \text{ whence, } x = \tfrac{3}{4}l.$$

The bending moment diagram is shown in Fig. 158(*d*).

Since the maximum negative moment ($-\tfrac{1}{8}wl^2$), which occurs at the wall, is greater than the maximum positive moment, the greatest unit-stress in this type of beam occurs at the wall and may be found from the flexure formula $\left(M = \dfrac{sI}{c}\right)$ by using $\tfrac{1}{8}wl^2$ for M.

Maximum Deflection.—Equation (94) is the equation of the elastic curve, the value of c_2 being zero and the value of R_1 being $\tfrac{3}{8}wl$. Thus, the elastic curve equation becomes

$$24EIy = \tfrac{3}{2}wl(x^3 - 3l^2x) - w(x^4 - 4l^3x).$$

Further, the maximum deflection is found (by equating (93) to zero) to occur at the distance $x = 0.4215l$ from the left support, and its value is

$$\Delta = 0.0054\frac{wl^4}{EI}.$$

As indicated in Fig. 158(*e*), the beam may be considered to be a combination of a simple beam having a span $\frac{3}{4}l$ and a cantilever of length $\frac{1}{4}l$. That is, the beam may be sawed in two and a hinge inserted at the section where the moment is zero (inflection point), the shear V' at this section acting as the reaction for the simple beam and as a concentrated load on the cantilever.

Alternative Method.—If the left support were removed, the beam would become a cantilever beam and the uniform load on the beam would cause the free end to deflect a distance $\frac{1}{8}\frac{wl^4}{EI}$ (Art. 46). But since the end does not deflect, the reaction R_1 must be a force which if acting alone on the end of the cantilever would cause an upward deflection equal to $\frac{1}{8}\frac{wl^4}{EI}$. Now the deflection due to a concentrated load P on the free end of a cantilever is $\frac{1}{3}\frac{Pl^3}{EI}$ (Prob. 212). Hence,

$$\frac{1}{3}\frac{R_1 l^3}{EI} = \frac{1}{8}\frac{wl^4}{EI}.$$

Therefore,

$$R_1 = \tfrac{3}{8}wl.$$

This method does not, however, avoid the use of the elastic curve equation since this equation was employed in determining the expressions used for the deflections.

PROBLEMS

239. Derive the expressions for the maximum positive moment, and the maximum negative moment, as given above, by considering the beam to be composed of two beams as shown in Fig. 158(*e*).

240. A steel I-beam is fixed at one end and supported at the other end, the length of span being 10 ft. It is loaded with a uniformly distributed load of 100 lb. per ft. Find (*a*) the point of inflection, (*b*) the maximum negative moment, (*c*) the maximum positive moment, (*d*) the section modulus of the beam using a working unit-stress of 16,000 lb. per sq. in. Also draw to scale the moment and shear diagrams.

Ans. (*d*) section modulus = 0.937 in.^{3} if weight of beam is neglected.

241. In Fig. 158(*c*) let the origin of axes be taken at the right-hand end of the beam (or let the beam be turned end for end so that the fixed end is the left-hand

end) and obtain from the elastic curve equation one equation involving V_0 and M_0; then write the two equilibrium equations involving V_0 and R_1 and solve for all three values.

242. A steel shaft 4 in. in diameter is used as a beam that is only partially restrained in a wall at one end and supported at the other end. The bending moment at the restrained end is $-\frac{1}{10}wl^2$. The beam is subjected to a uniformly distributed load w of 250 lb. per ft. of length and the clear span is 10 ft. Draw shear and moment diagrams for this beam, and determine the maximum bending unit-stress.

Ans. $s = 4780$ lb./in.^{2}

243. A timber beam 6 in. wide and 12 in. deep is used on a span of 16 ft. The beam is fixed at one end and supported at the other and is subjected to a uniformly distributed load. If the working unit-stress is 800 lb. per sq. in., what is the maximum load per foot of length the beam can resist. *Ans.* $w = 300$ lb./ft.

244. A 10-in. 25.4-lb. I-beam is supported at the left end and partially restrained in a wall at the right end, the clear span being 15 ft. The load including the weight of the beam is 1200 lb. per ft., and the reaction at the left end is $\frac{5}{12}wl$. Find the maximum fiber unit-stresses in the beam due to the negative and positive bending moments and indicate on a sketch where these stresses occur.

53. Beam Fixed at Both Ends; Uniform Load.—Fig. 159(*a*) represents such a beam, the length of span being l and the load per unit of length, w. Fig. 159(*b*) shows that the beam is held in equilibrium by a parallel force system in which there are four reactions, R_1, R_2, R_3 and R_4. But, as in the preceding article, it is desirable to replace these reactions by the resisting shears and resisting moments at the ends of the beams as indicated in Fig. 159(*c*), and hence the four unknown reactions are V_1, V_2, M_1 and M_2. Now from symmetry and from the equations of equilibrium we have:

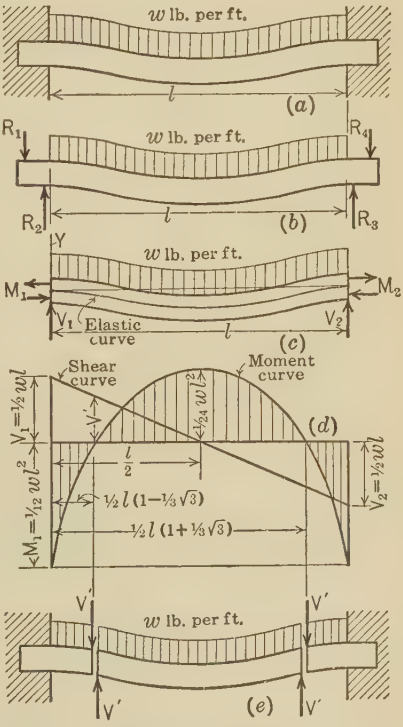

FIG. 159.—Beam fixed at both ends; load uniformly distributed.

$$M_1 = M_2 \quad \text{and} \quad V_1 = V_2 = \tfrac{1}{2}wl,$$

and hence only one [2] unknown reaction (namely, M_1 or M_2) need be found from the elastic curve equation. The value of M_1 may be found as follows:

If axes are chosen as shown in Fig. 159(c), the elastic curve equation is (see Art. 43)

$$EI\frac{d^2y}{dx^2} = M = -M_1 + V_1x - \frac{wx^2}{2}; \quad \ldots \quad (96)$$

M_1 is a negative moment since the top fibers are in tension. Integrating, we have

$$EI\frac{dy}{dx} = -M_1x + \frac{V_1x^2}{2} - \frac{wx^3}{6} + c_1,$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0, \text{ hence } c_1 = 0,$$

$$\frac{dy}{dx} = 0 \text{ when } x = l, \text{ hence}$$

$$0 = -M_1l + \tfrac{1}{2}wl\frac{l^2}{2} - \frac{wl^3}{6}.$$

Therefore, the magnitude of M_1 is $\frac{1}{12}wl^2$ and since it is a negative moment, the bending moment at the wall at each end of the beam is

$$M_1 = -\frac{1}{12}wl^2.$$

Shear and Moment Diagrams. Maximum Positive Moment.—The vertical shear for a section at the distance x from the left support is

$$V_x = \tfrac{1}{2}wl - wx;$$

this is equal to zero when $x = \frac{1}{2}l$, that is, at the mid-span. The shear diagram is shown in Fig. 159(d).

The bending moment about a section at a distance x from the left support is

$$M_x = -M_1 + V_1x - \frac{wx^2}{2}$$

$$= -\tfrac{1}{12}wl^2 + \tfrac{1}{2}wlx - \frac{wx^2}{2}.$$

[2] If the condition of symmetry were not employed, *two* unknowns (V_1 and M_1) would be found from the elastic curve equation, since only two equations of equilibrium are available.

But the bending moment is maximum at the section for which the vertical shear is zero and hence,

$$\text{Maximum Positive Moment} = -\tfrac{1}{12}wl^2 + \tfrac{1}{2}wl\frac{l}{2} - \frac{wl^2}{8}$$

$$= \frac{1}{24}wl^2.$$

Thus, the bending moment at the center of the beam is only one-half as large as that at the ends of the beam.

The points of inflection are found by equating the expression for the bending moment, M_x, to zero. Thus

$$M_x = -\tfrac{1}{12}wl^2 + \tfrac{1}{2}wlx - \frac{wx^2}{2} = 0,$$

whence

$$x = \tfrac{1}{2}l(1 \pm \tfrac{1}{3}\sqrt{3}).$$

The bending moment diagram is shown in Fig. 159(*d*).

Maximum Deflection.—The expression for the maximum deflection may be found from the elastic curve equation; if the axes are chosen as in Fig. 159(*c*), the elastic curve equation is,

$$EI\frac{d^2y}{dx^2} = M_x = -\tfrac{1}{12}wl^2 + \tfrac{1}{2}wlx - \frac{wx^2}{2}.$$

By integrating this equation twice, determining constants of integration, etc., the value of the maximum deflection is found to be,

$$\Delta = -\frac{1}{384}\frac{wl^4}{EI},$$

which is only one-fifth as large as that of a similarly loaded simple beam (see Art. 44).

It is sometimes convenient to consider the beam in question to be made up of two cantilevers and a simple beam as indicated in Fig. 159(*e*). The shears V' at the points of inflection are

$$V' = V_1 - wx$$

$$= \tfrac{1}{2}wl - w\tfrac{1}{2}l(1 - \tfrac{1}{3}\sqrt{3}).$$

PROBLEMS

245. Select a steel I-beam for a span of 12 ft. if the beam is to be fixed at both ends and subjected to a uniformly distributed load of 400 lb. per ft. Use a working unit-stress of 18,000 lb. per sq. in. Consult Table *A* in Appendix III.

Ans. 4-in. 10½-lb. I-beam.

246. Compare the maximum deflections of a timber beam having both ends fixed and a steel beam having simply supported ends. The beams are subjected to the same uniformly distributed load, and have the same span and dimensions. It is assumed that the unit-stress in the beams does not exceed the proportional limit of the material.

247. A beam with a clear span of 16 ft. is partially fixed at each end, the negative moment at each end being $\frac{1}{20}wl^2$. The load is uniformly distributed and equal to 1000 lb. per ft., including the weight of the beam. Find the size of the lightest steel I-beam that may be used without having the fiber unit-stress exceed 16,000 lb. per sq. in. Use Table A in Appendix III.

248. One of the beams in a theater balcony is 18 feet long. The left end is built into the wall in such a manner that the restraint produces a negative bending moment, M_0, at that end equal to 12,000 ft.-lb., and the beam rests on a support 3 feet from the right end. The beam is loaded with a uniformly distributed load of 600 lb. per ft. Determine the reactions on the beam, construct a shear diagram, and determine the value of the maximum positive bending moment. *Ans.* Max. pos. M = 9900 lb.-ft.

54. Beam Fixed at Both Ends; Concentrated Load at Mid-span.—Fig. 160(*a*) represents such a beam, the length of span being l and the load P. The reactions of the wall may be replaced by the shears V_1 and V_2 and the moments M_1 and M_2 (Fig. 160*b*) as was done in the preceding articles. These four unknown reactions may be found by use of the conditions of symmetry and equilibrium, and of the equation of the elastic curve of the left half of the beam. Thus, from the conditions of symmetry and equilibrium we have,

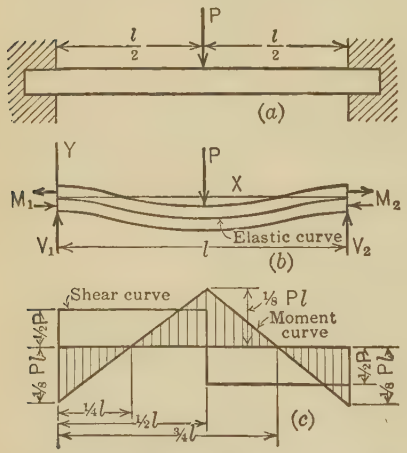

FIG. 160.—Beam fixed at both ends; concentrated load at mid-span.

$$M_1 = M_2 \quad \text{and} \quad V_1 = V_2 = \tfrac{1}{2}P.$$

Now M_1 may be found from the elastic curve equation; if axes are chosen as shown in Fig. 160(*b*) the equation of the elastic curve of the left half of the beam is

$$EI\frac{d^2y}{dx^2} = -M_1 + V_1x = -M_1 + \tfrac{1}{2}Px. \quad . \quad . \quad . \quad . \quad (97)$$

Integrating, we have

$$EI\frac{dy}{dx} = -M_1x + \frac{Px^2}{4} + c_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0, \text{ and hence } c_1 = 0.$$

$$\frac{dy}{dx} = 0 \text{ when } x = \frac{l}{2}, \text{ and hence,}$$

$$0 = -M_1\frac{l}{2} + \frac{Pl^2}{16}.$$

Therefore, the magnitude of M_1 is $\frac{1}{8}Pl$, and since it is a negative moment, the bending moment at the wall at each end of the beam is

$$M_1 = -\frac{1}{8}Pl.$$

The maximum positive moment occurs at the center where the vertical shear is zero (see Fig. 160c for shear diagram) and is

$$\text{Maximum Positive Moment} = -M_1 + V_1\frac{l}{2} = -\frac{1}{8}Pl + \frac{1}{2}P\frac{1}{2}l$$
$$= \frac{1}{8}Pl.$$

Thus the maximum positive moment is numerically equal to the maximum negative moment, and hence the maximum fiber unit-stresses at the wall and mid-span sections are equal, if the beam has a constant cross-section.

The inflection point in the curve of the left half of the beam is found, by equating the expression for the bending moment $(-\frac{1}{8}Pl + \frac{1}{2}Px)$ to zero, to be half way between the wall and the center of the span $(x = \frac{1}{4}l)$. The bending moment diagram is shown in Fig. 160(c).

By integrating equation (98) and observing that $y = 0$ when $x = 0$, and that $y = \Delta$ when $x = \frac{l}{2}$, the maximum deflection is found to be

$$\Delta = -\frac{1}{192}\frac{Pl^3}{EI},$$

which is only one-fourth as much as that of a simply supported beam similarly loaded (Art. 45).

PROBLEMS

249. A timber beam 6 in. wide and 12 in. deep is fixed at both ends and has a span of 16 ft. If it is subjected to a concentrated load at the center, what load can it safely resist if a working unit-stress of 800 lb. per sq. in. is used? *Ans.* $P = 4800$ lb.

250. Find the maximum deflection of the beam described in the preceding problem when the beam is subjected to a maximum fiber unit-stress of 800 lb. per sq. in.; $E = 1{,}500{,}000$ lb. per sq. in.

251. In Fig. 152 assume that P and l are known and determine M_0 and the shear and moment at the right-hand end of the beam, making use of the elastic curve equation. Also draw the shear and moment diagrams.

252. A 12-in. 55-lb. I-beam is used as a beam with both ends fixed. The beam has a clear span of 20 ft. and is subjected to a concentrated load of P lb. at the mid-span and a total uniformly distributed load of $2P$ lb. If the working fiber unit-stress is 16,000 lb./in.^{2}, what is the allowable value of P? Consult Table A in Appendix III. *Ans.* 12,160 lb.

55. Beam Fixed at One End, Supported at Other End, Concentrated Load at Mid-span.—Fig. 161(a) represents such a beam, the length of the span being l and the load, P. As in the preceding articles, the reactions exerted by the wall may be replaced by the resisting shear V_2 and the resisting moment M_2 acting on the section at the wall as shown in Fig. 161(b). Thus there are three unknown reactions (R_1, V_2, M_2) to be found. The left reaction R_1 may be found by use of the equations of the two elastic curves; the expressions for the bending moments on opposite sides of the load are different, and hence the elastic curves on opposite sides of the load are different; but the two curves have a common tangent and a common ordinate under the load. After R_1 has been found from the elastic curve equations, the two equations of equilibrium may be used to find the values of V_2 and M_2.

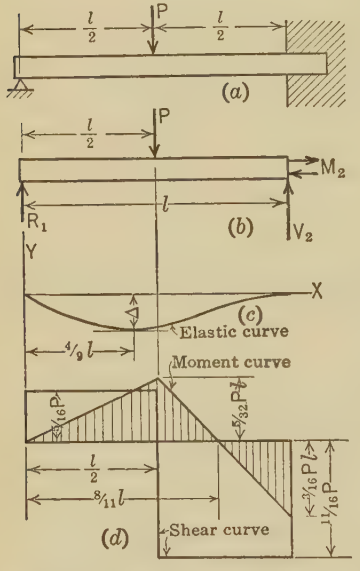

FIG. 161.—Beam fixed at one end, supported at other end; concentrated load at mid-span.

The procedure is as follows: Let the axes be chosen as shown in Fig. 161(c). The equations for the elastic curve of the left half of the beam are:

$$EI\frac{d^2y}{dx^2} = R_1x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

$$EI\frac{dy}{dx} = \tfrac{1}{2}R_1x^2 + c_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

$$EIy = \tfrac{1}{6}R_1x^3 + c_1x + c_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

The similar equations of the elastic curve of the right half are:

$$EI\frac{d^2y}{dx^2} = R_1x - P(x - \tfrac{1}{2}l). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

$$EI\frac{dy}{dx} = \tfrac{1}{2}R_1x^2 - \tfrac{1}{2}Px^2 + \tfrac{1}{2}Plx + c_2. \quad . \quad . \quad . \quad . \quad (103)$$

$$EIy = \tfrac{1}{6}R_1x^3 - \tfrac{1}{6}Px^3 + \tfrac{1}{4}Plx^2 + c_2x + c_4. \quad . \quad . \quad (104)$$

The five unknowns (c_1, c_3, c_2, c_4 and R_1) may be found by making use of the following conditions: In (101) $y = 0$ when $x = 0$; in (103) $\frac{dy}{dx} = 0$ when $x = l$; $\frac{dy}{dx}$ in (100) $= \frac{dy}{dx}$ in (103), when $x = \frac{1}{2}l$ in both (100) and (103); y in (101) $= y$ in (104), when $x = \frac{1}{2}l$ in both (101) and (104); and, in (104) $y = 0$ when $x = l$.

After the values of the constants are substituted in (101) and (104) the following equations of the elastic curves are found.

$$24EIy = 4R_1x^3 + 3Pl^2x - 12R_1l^2x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (101a)$$

$$48EIy = 8R_1x^3 - 8Px^3 + 12Plx^2 - 24R_1l^2x + Pl^3, \quad (104a)$$

and by using the last of the conditions stated above, namely, $y = 0$ in (104) when $x = l$, the value of R_1 is found to be

$$R_1 = \tfrac{5}{16}P.$$

Now from the two equations of equilibrium we have (see Fig. 161*b*),

$$\Sigma F = \tfrac{5}{16}P + V_2 - P = 0; \qquad \text{whence, } V_2 = \tfrac{11}{16}P.$$

$$\Sigma M = \tfrac{5}{16}Pl - P\tfrac{1}{2}l + M_2 = 0; \text{ whence, } M_2 = \tfrac{3}{16}Pl.$$

The vertical shear and bending moment at the wall, however, are equal *but opposite* to the reactions V_2 and M_2 and are negative, the vertical shear being downward and the bending moment causing tension on the top side of the beam. Hence if V_2 and M_2 denote the vertical shear and bending moment for the section at the wall, we have

$$V_2 = -\tfrac{11}{16}P \quad \text{and} \quad M_2 = -\tfrac{3}{16}Pl \text{ (as indicated in Fig. 161}d\text{).}$$

The reader should also obtain these values by simply writing the expressions for V_2 and M_2 in terms of the forces to the left of the section.

The maximum positive moment occurs at the mid-span and is,

$$\text{Maximum Positive Moment} = \tfrac{5}{16}P \times \tfrac{1}{2}l = \tfrac{5}{32}Pl.$$

The shear and moment diagrams are shown in Fig. 161(*d*).

56. Comparison with Simple Beams.—In the preceding articles it has been shown that a fixed-ended beam is stronger and stiffer than a similar beam that is simply supported at its ends. This is true because the positive bending moment in the central portion of a beam is decreased due to the negative moment at the ends, and thus more of the material of a fixed beam is effective in resisting the loads than in the simply supported beam, since the bending moment is more nearly uniform along the length of the fixed beam.

The effect of applying negative moments at the ends of a simply supported beam is shown in Fig. 162. The curve *CED*, referred to *CD* as a base line, represents the bending moment diagram for a simply supported beam subjected to a uniformly distributed load of w lb. per ft., the maximum bending moment (as shown in Art. 33, Prob. 152) is at the mid-span and is equal to $\frac{1}{8}wl^2$. Now if the beam were a fixed-ended beam the negative moments at the ends would be $\frac{1}{12}wl^2$ (as shown in Art. 53) and the curve *CED* would still represent the moment diagram if *AB*, Fig. 162(*b*), instead of *CD*, represents the axis or base line. Thus the moment at the center is decreased by the amount of the negative moment at the end. Generalizing, *in a beam fixed at both ends, the sum of the maximum negative and positive moments is equal to the maximum moment in a similarly loaded simple beam.* The student should show that this statement is true if the load is concentrated at the mid-span.

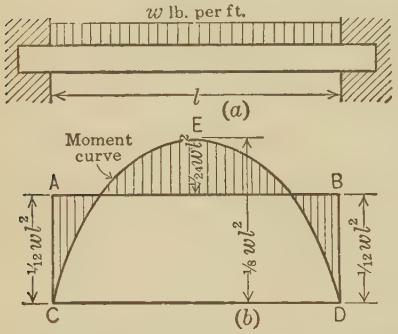

Fig. 162.—Relation between moment diagrams for fixed and simple beams.

In practice, however, it frequently is difficult to determine to what extent a beam is restrained at the ends, since yielding of the end-restraining bodies such as abutments, riveted end-connections, etc., decreases the negative moments at the ends and increases the positive moment by an equal amount. Likewise, unequal settlements of the ends and temperature changes influence the stresses in fixed-ended beams to a greater extent than in simply supported beams. For these reasons fixed-ended beams are not used in preference to simple beams as extensively as the above results would seem to justify. However, test results[3] obtained with certain types of riveted end-connections show

[3] Tests to Determine the Rigidity of Riveted Joints of Steel Structures. Bull. No. 104, Engineering Experiment Station, University of Illinois.

that the moments at the ends are reduced but little by the yielding of these riveted end-restraining connections.

57. Continuous Beams. Theorem of Three Moments.—A continuous beam is one that rests on more than two supports. Since the number of reactions is greater than two, use is made of the elastic curve equation in order to obtain the reactions of the supports, the two equations of equilibrium alone being insufficient.

However, instead of determining the reactions first and then finding the bending moment (and hence fiber unit-stress) at any section, it is more convenient to determine first the negative moments at sections over all the supports and then to find from these negative moments the reactions of the supports and the bending moments and vertical shears at other sections. The negative moments may be found by use of the

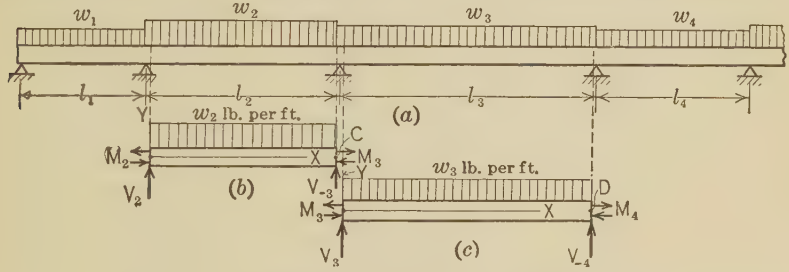

FIG. 163.—Moments at supports of continuous beam.

theorem of three moments which may be derived from the elastic curve equations of two adjacent spans as follows:

Theorem of Three Moments, Load Uniformly Distributed.—Fig. 163(*a*) represents a continuous beam of several unequal spans; each span is subjected to a uniformly distributed load, but the load is not constant over the whole length of the beam; all supports are assumed to be on the same level. Consider two adjacent spans (second and third spans) in Fig. 163. Let M_2, M_3, M_4, etc., denote the bending moments at the second, third, fourth, etc., supports. Let V_2 denote the shear on a section just to the right of the second support and V_{-2} just to the left of the second support; similarly let V_3 and V_{-3} denote the shears on the right and left, respectively, of the third support, etc.

A free-body diagram showing all the forces acting on that part of the beam between the second and third supports is shown in Fig. 163(*b*) and a similar diagram for the part of the beam between the third and fourth supports is shown in Fig. 163(*c*).

For the second span let the axes be chosen as indicated in Fig. 163(*b*). Then, the elastic curve equation for the second span may be written as

follows, in which the sign of M_2 is assumed to be positive even though it is known to be negative as shown in Fig. 163(b), (the solution of the equations will determine the correct sign),

$$EI\frac{d^2y}{dx^2} = M_2 + V_2x - \frac{w_2x^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

$$EI\frac{dy}{dx} = M_2x + \frac{V_2x^2}{2} - \frac{w_2x^3}{6} + c_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

$$EIy = \frac{M_2x^2}{2} + \frac{V_2x^3}{6} - \frac{w_2x^4}{24} + c_1x + c_2. \quad . \quad . \quad (107)$$

Similarly for the third span let the origin be chosen at the third support as shown in Fig. 163(c); the sign of M_3 is taken as positive in the following equations as was done for M_2 in the above equations.

$$EI\frac{d^2y}{dx^2} = M_3 + V_3x - \frac{w_3x^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

$$EI\frac{dy}{dx} = M_3x + \frac{V_3x^2}{2} - \frac{w_3x^3}{6} + c_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

$$EIy = \frac{M_3x^2}{2} + \frac{V_3x^3}{6} - \frac{w_3x^4}{24} + c_3x + c_4. \quad . \quad . \quad (110)$$

Now in (107), $y = 0$ when $x = 0$, and hence $c_2 = 0$; also $y = 0$ when $x = l_2$ and hence the value of c_1 may easily be obtained. Again, in (110) $y = 0$ when $x = 0$, and hence $c_4 = 0$; also $y = 0$ when $x = l_3$ and hence the value of c_3 may be obtained. Further $\frac{dy}{dx}$ in (106) is equal to $\frac{dy}{dx}$ in (109) when x in (106) is equal to l_2 and x in (109) is equal to zero. By making use of these facts the following equation is found.

$$12M_2l_2 + 8V_2l_2^2 - 3w_2l_2^3 = -12M_3l_3 - 4V_3l_3^2 + w_3l_3^3. \quad (111)$$

Now V_2 and V_3 may be expressed in terms of M_2, M_3, M_4 and the known quantities l_2, l_3, w_2 and w_3 by using one of the equations of equilibrium for the forces in Fig. 163(b) and in Fig. 163(c). Thus, for the second span (Fig. 163b) one equilibrium equation is (M_3 in Fig. 163(b) is opposite to M_3 in Fig. 163(c) and hence is negative)

Hence,

$$\left.\begin{aligned} \Sigma M_C &= M_2 + V_2l_2 - \tfrac{1}{2}w_2l_2^2 - M_3 = 0. \\ V_2 &= \frac{-M_2 + \frac{1}{2}w_2l_2^2 + M_3}{l_2} \end{aligned}\right\}, \quad . \quad . \quad (112)$$

and for the third span,

$$\Sigma M_D = M_3 + V_3 l_3 - \tfrac{1}{2} w_3 l_3^2 - M_4 = 0.$$

Hence,

$$V_3 = \frac{-M_3 + \tfrac{1}{2} w_3 l_3^2 + M_4}{l_3} \quad \Bigg\}, \quad . \quad . \quad (113)$$

in which all the moments are assumed to be positive.

By substituting these values of V_2 and V_3 in equation (111) the following relation between the bending moments at three consecutive supports is found:

$$M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = -\tfrac{1}{4} w_2 l_2^3 - \tfrac{1}{4} w_3 l_3^3,$$

or, if the subscripts 1, 2, and 3 are used to refer to any three consecutive supports and spans the equation may be written,

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\tfrac{1}{4} w_1 l_1^3 - \tfrac{1}{4} w_2 l_2^3, \quad . \quad (114)$$

which expresses the *theorem of three moments* for a continuous beam subjected to uniformly distributed loads and resting on supports on the same level.

If the spans are equal ($l_1 = l_2 = l$) and each span carries the same load ($w_2 = w_3 = w$) the equation becomes

$$M_1 + 4M_2 + M_3 = -\tfrac{1}{2} w l^2, \quad . \quad . \quad . \quad . \quad . \quad (115)$$

which expresses the theorem of three moments for a continuous beam with equal spans and with a constant uniformly distributed load over the entire beam.

58. Solution of Typical Problem.—Let it be required to draw the shear and moment diagrams for a continuous beam of four spans, each of length l, when subjected to a load of w lb. per ft. over the entire beam, and also to find the maximum fiber unit-stress and the equation of the elastic curve of the first span. (See Fig. 164*a*) The known quantities are: w, l, E and I. The main steps in the solution are:

(*a*) By use of the theorem of three moments find the negative bending moment over all supports.

(*b*) Find the reactions of the supports from the negative moments over the supports; all the external forces acting on the beam will then be known.

(*c*) Find the vertical shears at various sections and draw the shear diagram.

(*d*) Find the bending moments at various sections and draw the moment diagram.

(*e*) Equate the maximum bending moment to $\frac{sI}{c}$ and solve for s. the maximum fiber unit-stress.

(*f*) In the elastic curve equation $EI\frac{d^2y}{dx^2} = M$ substitute for M the expression for the bending moment in the first span and integrate twice, etc.

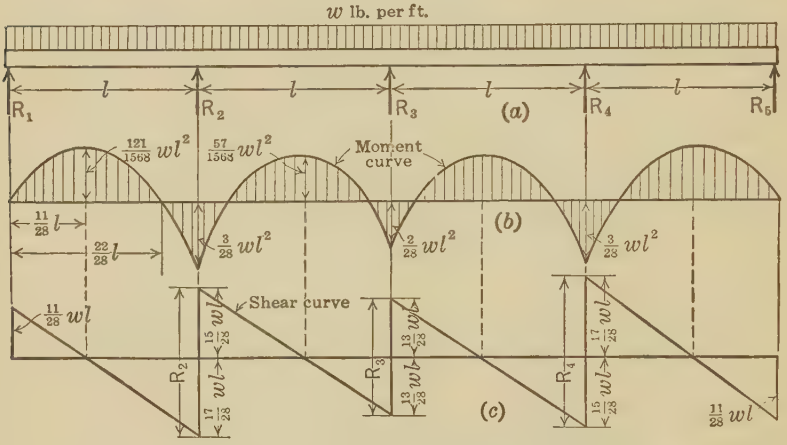

FIG. 164.—Moment and shear diagrams for continuous beam.

Solution.—(*a*) In accordance with the theorem of three moments, the following equations may be written:

$$M_1 + 4M_2 + M_3 = -\tfrac{1}{2}wl^2,$$

$$M_2 + 4M_3 + M_4 = -\tfrac{1}{2}wl^2,$$

$$M_3 + 4M_4 + M_5 = -\tfrac{1}{2}wl^2.$$

Further, $M_1 = M_5 = 0$ since there is no restraint at the ends; and $M_2 = M_4$, from the condition of symmetry. The solution of these equations gives the following values:

$$M_1 = 0,\ M_2 = -\tfrac{3}{28}wl^2,\ M_3 = -\tfrac{2}{28}wl^2,\ M_4 = -\tfrac{3}{28}wl^2,\ M_5 = 0.$$

(*b*) Now the bending moment at a section over the second support as found above is $-\frac{3}{28}wl^2$, but the bending moment at any section is the algebraic sum of the moments of the forces to the left of the section. Hence,

$$R_1 l - wl\frac{l}{2} = -\frac{3}{28}wl^2;\ R_1 = \frac{11}{28}wl.$$

Similarly for the moment over the third support we have,

$$R_1 \cdot 2l + R_2 l - w \cdot 2l \cdot l = -\tfrac{2}{28}wl^2;\ R_2 = \tfrac{32}{28}wl.$$

In a similar way the value of R_3 is found to be, $R_3 = \frac{2}{2}\frac{6}{8}wl$, and from symmetry, $R_4 = R_2$ and $R_5 = R_1$.

(*c*) The shear and moment diagrams are shown in Fig. 164(*b*) and (*c*). The vertical shear at any section in the first span at the distance x from the left support is

$$V_x = R_1 - wx = \tfrac{1}{2}\tfrac{1}{8}wl - wx,$$

and hence the vertical shear is zero when $x = \frac{1}{2}\frac{1}{8}l$; the maximum positive moment occurs at this section.

The shear just to the left of the second support is

$$V_{-2} = R_1 - wl = \tfrac{1}{2}\tfrac{1}{8}wl - wl = -\tfrac{1}{2}\tfrac{7}{8}wl,$$

and the shear just to the right of the second support is

$$V_2 = R_1 + R_2 - wl = \tfrac{1}{2}\tfrac{1}{8}wl + \tfrac{3}{2}\tfrac{2}{8}wl - wl = +\tfrac{1}{2}\tfrac{5}{8}wl.$$

Thus, the shear changes at the second support from $-\frac{1}{2}\frac{7}{8}wl$ to $+\frac{1}{2}\frac{5}{8}wl$ due to the reaction R_2; in other words, the reaction of a support is equal to the arithmetic sum of the shears at the two sides of the support.

The shears in other spans may be found by the same method as used above for the first span; the values are given in Fig. 164(*b*).

Thus the reaction at any support may be found by solving for the shears on the two sides of the support and adding them rather than by the method used under (*b*) above.

(*d*) The bending moment at any section in the first span is

$$M'_x = R_1 x - \frac{wx^2}{2},$$

and as noted above this is maximum when $x = \frac{1}{2}\frac{1}{8}l$. Therefore, the maximum positive moment is

$$\text{Max. pos. moment} = \frac{11}{28}wl \cdot \frac{11}{28}l - \frac{w}{2}\left(\frac{11}{28}l\right)^2 = \frac{121}{1568}wl^2,$$

which is less than the maximum negative moment ($\frac{3}{2}\frac{}{8}wl^2$).

The inflection point occurs where $M'_x = 0$; thus

$$M'_x = \frac{11}{28}wl \cdot x - \frac{wx^2}{2} = 0, \quad \text{whence} \quad x = \tfrac{2}{2}\tfrac{2}{8}l.$$

The bending moment at any section in the second span is

$$M''_x = M_2 + V_2 x - \tfrac{1}{2}wx^2,$$

$$= -\tfrac{3}{2}\tfrac{}{8}wl^2 + \tfrac{1}{2}\tfrac{5}{8}wlx - \tfrac{1}{2}wx^2.$$

The moment is maximum when $x = \frac{1}{2}\frac{5}{8}l$, since this is the value that makes the shear in the second span equal to zero. The value of the maximum positive moment in this span is $\frac{5}{1}\frac{7}{5}\frac{}{6}\frac{}{8}wl^2$.

(*e*) The greatest bending moment that the beam is subjected to is the negative moment at the second (or fourth) support, and hence the maximum unit-stress in the beam is

$$s = \frac{Mc}{I} = \frac{3}{28}wl^2 \cdot \frac{c}{I}.$$

(*f*) The elastic curve equation for the first span is

$$EI\frac{d^2y}{dx^2} = R_1x - \frac{wx^2}{2} = \frac{11}{28}wlx - \frac{wx^2}{2}.$$

Integrating, we have,

$$EI\frac{dy}{dx} = \frac{11}{28}wl\frac{x^2}{2} - \frac{wx^3}{6} + c_1$$

$$EIy = \frac{11}{28}wl\frac{x^3}{6} - \frac{wx^4}{24} + c_1x + c_2.$$

Now, when $x = 0$, $y = 0$; and when $x = l$, $y = 0$; thus $c_2 = 0$, and c_1 may easily be found to be $-\frac{1}{42}wl^3$. Therefore the elastic curve equation of the first span is

$$EIy = \frac{11}{28}wl\frac{x^3}{6} - \frac{wx^4}{24} - \frac{1}{42}wl^3x.$$

59. Values of Moments and Shears.—In Fig. 165 are given the values of the coefficients from which the negative moments over the supports and the vertical shears at either side of each support may be found for continuous beams resting on supports on the same level, having equal spans of l ft. each, and carrying a constant uniform load of w per unit length on each span.

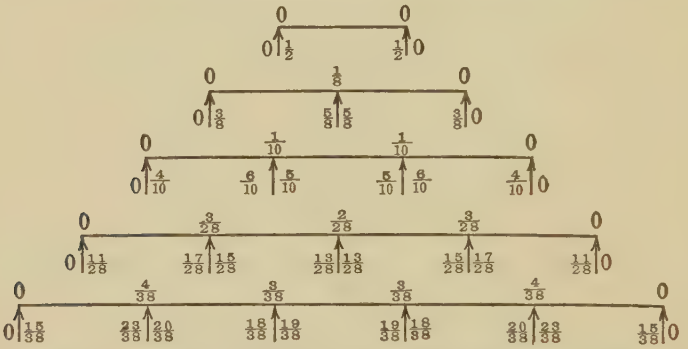

FIG. 165.—Values of shear and moment coefficients for continuous beam of equal spans subjected to uniform load.

The values directly above the supports are the moment coefficients, that is, the numbers by which $-wl^2$ must be multiplied to obtain the bending moments over the supports. Similarly, the numbers on either side of the supports are the shear coefficients, that is, the numbers by which wl must be multiplied to obtain the magnitude of the vertical shear for the section; the vertical shear is negative for the section to the left of each support and positive for the section to the right of each support. The reaction at each support is the arithmetic sum of the

vertical shears on the two sides of the support. The values given in Fig. 165 may be found by the methods discussed in Art. 58.

PROBLEMS

253. A continuous beam consists of four equal spans each 12 ft. long, and is subjected to a uniform load of 100 lb. per ft. over its entire length. (*a*) Find the reaction of the second support. (*b*) If the beam has a rectangular cross-section, the depth being twice the width, what is the depth if the maximum fiber unit-stress in the beam is 1000 lb. per sq. in.? *Ans.* (*a*) $R_2 = 1370$ lb., (*b*) depth = 6.06 in.

254. A 12-in. 55-lb. I-beam is used as a horizontal continuous beam over four spans. The lengths of the spans are 12, 16, 16, and 12 ft. respectively, from left to right. The loads are uniformly distributed over each span, the loads per foot of length on the four spans being 1600, 2400, 2400, and 1600 lb. per ft., respectively. Find the maximum fiber unit-stress in the beam.

Ans. $M_3 = -56{,}800$ lb.-ft. $s = 12{,}800$ lb. per sq. in.

255. A continuous beam has three spans of 10 ft. each. The first span is subjected to a uniformly distributed load of 3000 lb. per ft. The other spans are not loaded. Find the four reactions.

Ans. $R_1 = 13{,}000$ lb. $R_2 = 19{,}500$ lb. $R_3 = -3000$ lb. $R_4 = 500$ lb.

256. What should be the depth of a pine continuous beam having four equal spans of 7 ft. in order to resist a uniformly distributed load of 300 lb. per ft. over each span? Use a working unit-stress of 1000 lb. per sq. in. and make the depth of the beam twice the width.

257. An 8-in. 25.5-lb. I-beam is continuous over three spans, each having a length of 16 ft. The first and third spans are loaded with 800 lb. per ft. and the second span is not loaded. Find the maximum fiber unit-stress in the beam and draw, to scale, the moment and shear diagrams.

258. A beam is continuous over three spans of 20, 10, and 15 ft. respectively. The first and third spans are each loaded with a uniformly distributed load of 400 lb. per ft. and the second span is not loaded. Find the moment over each support and the shear at the right of the first support.

259. A continuous beam of two spans, 20 ft. and 18 ft., carries a uniformly distributed load of 1000 lb. per ft. over the shorter span and no load on the other span. Find the maximum positive and negative bending moments and the reactions at all three supports.

260. A continuous beam 30 ft. long is attached to supports at each end and at points 10 ft. and 20 ft. from the left end. The center space carries a uniformly distributed load of 1200 lb. per ft. Determine the maximum positive and negative bending moments and the reactions at all the supports.

261. A beam 40 ft. long is attached to a support at its center and rests on supports 10 ft. from each end; that is, each end overhangs 10 ft. Each overhanging end carries a uniformly distributed load of 200 lb. per ft. Find the bending moment and the reaction at each of the three supports and plot the shear and moment diagrams. *Ans.* $M_2 = +5000$ lb.-ft.; $R_1 = R_3 = +3500$ lb.; $R_2 = -3000$ lb.

60. Advantages and Disadvantages of Continuous Beams.—The remarks in Art. 56 concerning fixed-ended beams apply also, in the main, to continuous beams; the negative moments over the supports reduce the moments (and hence stresses) near the centers of the spans, and hence continuous beams are stronger and stiffer than simply supported beams of equal spans. On the other hand, the uneven settlement of supports may change the moments at the supports and throughout the beam from those found by the analysis given above; further, the stiffer the beams the greater is the change in the moments. Partly for this reason and partly because the loads on the various spans of continuous beams in structures may vary considerably from that assumed in the design, it is frequently assumed in practice that in uniformly loaded beams of four or more equal spans all the spans (except the end spans), are subjected to the same maximum bending moment, a common value being $\frac{1}{12}wl^2$; the maximum moment in the end spans is somewhat greater ($\frac{1}{10}wl^2$ is frequently used). Compare these values with those given in Fig. 165 for beams having more than four spans and note that the moments do not vary greatly over the inner supports. Two-span and three-span beams are stressed higher than beams of four or more spans, and for such beams the maximum moment in each span is frequently assumed to be $\frac{1}{10}wl^2$. However, the values used in any case may be based on the results of an analysis of the moments in continuous beams similar to that discussed in the preceding articles.

61. Theorem of Three Moments for Concentrated Loads.—The loading on most continuous beams may be assumed to be uniformly distributed without introducing serious errors, but in some cases heavy concentrated loads act on the beams and the results given in Art. 59 are not applicable. Space does not permit the derivation, but the theorem of three moments that applies to a beam of constant cross-section with supports on the same level and subjected to concentrated loads is as follows:

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2$$

$$= - P_1l_1{}^2(k_1 - k_1{}^3) - P_2l_2{}^2(2k_2 - 3k_2{}^2 + k_2{}^3). \quad . \quad (116)$$

M_1, M_2 and M_3 are the bending moments at any three consecutive supports, l_1 and l_2 being the lengths of the two consecutive spans at the ends of which the moments are M_1, M_2 and M_3. P_1 is a concentrated load in the first of the two spans at the distance k_1l_1 from the first of the three supports, and P_2 a concentrated load in the second of the two spans at the distance k_2l_2 from the second of the three supports, where k_1 and k_2 are proportional parts of the spans such as $\frac{1}{3}$, $\frac{1}{2}$, etc.

PROBLEM

262. A continuous beam has two spans, the left span 20 ft. long and the right span 18 ft. long. The 20-ft. span carries a load of 16,000 lb. 10 ft. from the left end, and the 18-ft. span carries a load of 15,000 lb. 8 ft. from the right end. Find the proper size of a steel I-beam to resist these loads using a working unit-stress of 16,000 lb. per sq. in. *Ans.* $M_2 = -$ 54,300 lb.-ft.; 12-in. 40.8-lb. I-beam.

REVIEW QUESTIONS AND PROBLEMS

263. A beam is said to be "fixed" at any section when the slope to the elastic curve at the section is zero $\left(\frac{dy}{dx} = 0\right)$. From the results found in Art. 52 determine the value of P in Fig. 166 so that the beam will be fixed at the section at the right-hand support. Neglect the weight of the beam. *Ans.* $P = \frac{1}{2}wl$.

264. What use is made of the elastic curve equation beside that of finding the deflection of a beam?

265. Find, by using the elastic curve equation and *one* equilibrium equation, the reactions of a simple beam of span l carrying a uniformly distributed load of w per unit length; that is, show that each reaction is $\frac{1}{2}wl$. Observe that much less work is involved by applying the *two* equilibrium equations, and thus note why the elastic curve equation is used for determining reactions only when equilibrium equations are insufficient, that is for statically indeterminate beams.

266. A steel I-beam is fixed at each end and is subjected to a uniformly distributed load of 600 lb. per ft. of span. The span is 12 ft. Calculate the maximum bending moment in the beam and state where this moment occurs. If the working tensile and compressive unit-stress is 16,000 lb. per sq. in., what section modulus should the beam have? *Ans.* 5.4 in.^{3}

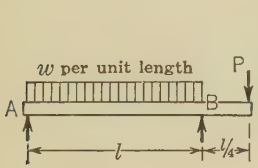

FIG. 166.

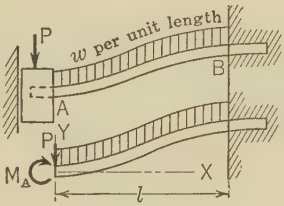

FIG. 167.

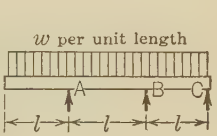

FIG. 168.

267. Find, by using the elastic curve equation, the bending moment M_A at the section A of the beam shown in Fig. 167 in terms of the known quantities P, l, and w. The beam is fixed at both ends. *Ans.* $M_A = (\frac{1}{2}Pl + \frac{1}{6}wl^2)$.

268. Find the bending moment at each support of the beam shown in Fig. 168 and the reaction of each support. Also draw the shear and moment diagrams.

Ans. $M_A = -\frac{wl^2}{2}$, $M_B = 0$, $R_A = 2wl$, $R_B = \frac{1}{2}wl$, $R_C = \frac{1}{2}wl$.

CHAPTER VII

COMBINED AXIAL AND BENDING LOADS. ECCENTRIC LOADS

62. Introduction.—In the preceding chapters stresses caused by axial, torsional, and bending loads were found when a member was subjected to loads of one type only. Members of many structures and machines, however, are subjected to loads of two or more of these types (or the actual loads may conveniently be resolved into loads of two or more of these types), and the unit-stress developed at any point in the member may frequently be found by assuming that the loads act independently and hence each load is assumed to produce the same unit-stress that it would produce if it were the only load acting on the member; these unit-stresses may then be combined, if the stresses are within the proportional limit, to obtain the actual unit-stress. This principle, called the principle of superposition (see last paragraph in Art. 43), will be used in this chapter in determining the normal (tensile or compressive) unit-stress at a point in a member when the member is subjected to combined axial (tensile or compressive) and bending loads, and also in determining the shearing unit-stress when the member is subjected simultaneously to a central shearing load and torsional loads.

63. A Beam Subjected to an Axial End Load.—In Fig. 169(a) is represented a simple beam of length l feet subjected to a uniformly distributed transverse load of w pounds per foot and a compressive axial load of Q pounds at its ends. Let it be required to find the maximum normal (tensile or compressive) unit-stress on a transverse cross-section of the beam. The beam will be assumed to be short so that the deflection of the beam may be neglected without introducing serious error in the results, and hence the load Q may be assumed to be an axial load for each cross-section of the beam. Further, the weight of the beam is assumed to be negligible in comparison with the other loads on the beam.

The maximum axial unit-stress will occur at the mid-section since the unit-stress due to the transverse bending load is maximum at the mid-section and the direct stress due to Q is the same for all sections. Now as indicated by the free body diagram of the left half of the beam in Fig. 169(b), if Q were the only force acting, the stress would be uniformly distributed on the cross-sectional area a, the unit-stress at any

point in the area being s_1. The total stress would be as_1 and would be equal and opposite to Q since it would hold Q in equilibrium. Hence,

$$Q = as_1 \quad \text{or} \quad s_1 = \frac{Q}{a}.$$

If, on the other hand, the bending loads were the only forces acting, the bending moment M at the mid-section (equal to $\frac{1}{8}wl^2$, Fig. 103) would develop the resisting moment $\frac{s_2 I}{c}$ as shown in Fig. 169(b), the fiber unit-stress s_2 varying directly as the distance of the fiber from the neutral axis provided that its value does not exceed the proportional limit of the material.[1] And since the resisting moment holds the bending

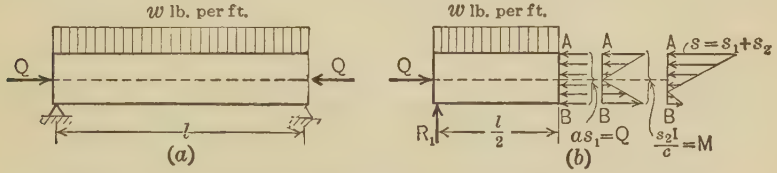

FIG. 169.—Stress in beam subjected to end-loads; deflection neglected.

moment in equilibrium, the two moments are numerically equal. Hence,

$$M = \frac{s_2 I}{c} \quad \text{or} \quad s_2 = \frac{Mc}{I}.$$

The normal unit-stress at any point of the mid-section then, according to the principle of superposition, is the algebraic sum of the unit-stresses caused by the loads acting separately; the maximum unit-stress is a compressive stress and occurs on the top fiber; its value is

$$s = s_1 + s_2 = \frac{Q}{a} + \frac{Mc}{I}.$$

The unit-stress on the bottom fiber may be either tensile or compressive according as s_2 is larger or smaller than s_1. As shown in Fig. 169(b), s_2 is larger than s_1, therefore, the unit-stress on the bottom fiber is a tensile stress, and the surface of zero fiber-stress is some distance below the centroidal plane of the beam. Further, if the load Q were a tensile load the maximum unit-stress would occur on the bottom fiber and would be a tensile stress.

[1] For other limitations and assumptions see Art. 41.

Deflection of Beam Not Negligible.—If the deflection of the beam is not negligible, the load Q (Fig. 170a) cannot be considered to be an axial load with respect to the mid-section (or any other cross-section except the end sections). Let Δ denote the deflection of the axis of the beam at the section on which the stress is to be found (in this problem the mid-section). Now if two equal opposite and collinear forces Q_1, Q_2 (each equal to Q), are applied to the beam as shown in Fig. 170(b), the force Q will be resolved into a force Q_1, which is axial with respect to the mid-section, and a couple having a moment $Q\Delta$; these two forces do not change the stresses developed on the mid-section, but simply modify the force system so that the equations $\left(P = as \text{ and } M = \frac{sI}{c}\right)$ already developed in the preceding chapters can be made to apply to this problem. Thus, since Q_1 passes through the centroid of the mid-section,

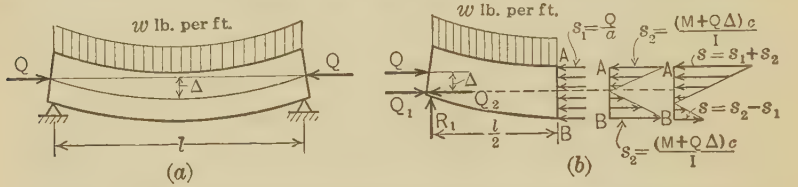

FIG. 170.—Stress in beam subjected to end-loads; deflection considered.

the unit-stress s_1 due to Q_1 alone is constant over the section and is obtained from the equation

$$Q_1 = Q = as_1,$$

as indicated in Fig. 170(b). The other forces constitute a bending moment composed of the cross-bending moment M, as in the previous problem, and the moment $Q\Delta$. The total bending moment is therefore, $M + Q\Delta$ and is held in equilibrium by the resisting moment $\frac{s_2 I}{c}$. Therefore,

$$M + Q\Delta = \frac{s_2 I}{c},$$

as indicated in Fig. 170(b). Thus the maximum unit-stress s is,

$$s = s_1 + s_2$$

$$= \frac{Q}{a} + (M + Q\Delta)\frac{c}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

But since the value of Δ depends on the value of the total bending moment $(M + Q\Delta)$, and the value of the bending moment depends in

turn on the value of Δ, a method of approximation in solving equation (117) is, as a rule, the most convenient method. Thus, the value of Δ that would be caused by the cross-bending moment, M, alone is found first $\left(\text{equal in this problem to } \frac{5}{384}\frac{wl^4}{EI}, \text{ Art. } 44\right)$; this value of Δ is then used in the expression $M + Q\Delta$ and this new value of the total bending moment is used to find a closer approximation to the value of Δ. This operation may be repeated as many times as desired.

ILLUSTRATIVE PROBLEM

Problem 269.—In Fig. 171(a) is shown a wall bracket, the horizontal boom BC being a 7-in. 15.3-lb. I-beam. The beam is pin-connected to the post DE at B and to the rod AC at C. If the load P is 4900 lb. find the maximum fiber unit-stress in the I-beam, assuming that the weight of the beam is negligible.

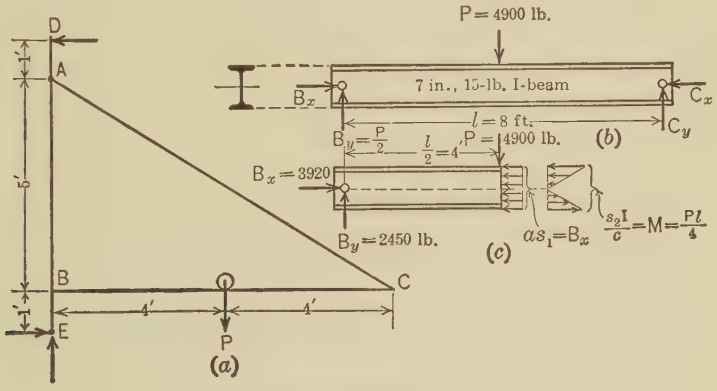

FIG. 171.—Stress in boom of wall crane.

Solution.—Deflection Neglected.—The forces acting on the I-beam are shown in Fig. 171(b). The normal stresses at the mid-section due to the end load B_x and the cross-bending loads are shown in Fig. 171(c). The value of B_x and of C_x is equal to the horizontal component (x-component) of the tension in the tie rod AC, and this tension may be found by applying the conditions of equilibrium either algebraically or graphically to the forces acting on the pin at C or on the post DE; it is assumed that the student is familiar with methods of finding the forces acting on the members of pin-connected structures. The value of the tension in the rod is found to be 4600 lb. and its x-component is 3920 lb. Therefore B_x equals 3920 lb. Further, from a steel-maker's handbook the area a of the cross-section of the I-beam is found to be 4.43 sq. in. and the section modulus I/c is equal to 10.4 in.^{3}

As indicated in Fig. 171(c) the compressive unit-stress, s, at the top of the I-beam is

$$s = s_1 + s_2$$

$$= \frac{B_x}{a} + \frac{Pl}{4}\cdot\frac{c}{I} = \frac{3920}{4.43} + \frac{4900 \times 8 \times 12}{4} \times \frac{1}{10.4}$$

$$= 885 + 11{,}300 = 12{,}200 \text{ lb. per sq. in.}$$

Deflection Not Neglected.—The deflection Δ caused by the transverse forces alone is (Art. 45)

$$\Delta = \frac{1}{48}\frac{Pl^3}{EI} = \frac{1}{48}\frac{4900(8\cdot 12)^3}{30{,}000{,}000\cdot 36.2} = 0.0832 \text{ in.}$$

When the beam deflects, the load B_x has a moment $B_x\Delta$ with respect to the mid-section and, as explained above, the bending moment is $M + B_x\Delta$. Thus the compressive unit-stress on the top fiber of the beam is

$$s = s_1 + s_2 = \frac{B_x}{a} + \left(\frac{Pl}{4} + B_x\Delta\right)\frac{c}{I}$$

$$= 885 + 11{,}300 + 3920\cdot 0.0832\cdot\frac{1}{10.4}$$

$$= 885 + 11{,}300 + 31.3.$$

Thus, very little error is introduced in this problem by neglecting the deflection of the beam.

64. Eccentric Longitudinal Load in Plane of Symmetry.—In Fig. 172(*a*) is represented a short [2] compression member acted on by an eccentric load P in a plane containing an axis of symmetry [3] of each cross-section of the member, the amount of the eccentricity being denoted by e. Let it be required to find the maximum unit-stress developed on any normal cross-section, a, of the member.

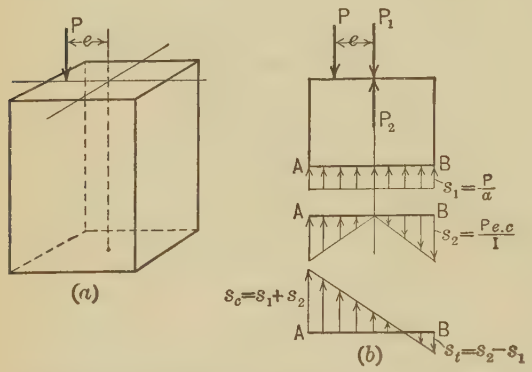

FIG. 172.—Stress due to eccentric load in plane of symmetry.

The force P may be resolved into an axial load P_1 (equal to P) and a couple having a moment Pe (or P_2e since $P_1 = P_2 = P$) without affecting the stresses developed on the section (Fig. 172*b*). Now if P_1 were acting alone it would cause a constant unit-stress s_1 on the area a such that,

$$P_1 = P = as_1.$$

[2] The effect of an eccentric load on a *long* compression member (column) in which the deflection of the member must be considered, is discussed in Art. 82.

[3] The stress caused by an eccentric load that does not act at a point on an axis of symmetry or principal axis, is discussed in Art. 139. All axes, however, are principal axes for a member having a square or circular cross-section; that is, the moments of inertia with respect to all axes in the plane of the area passing through the centroid of a square or circular area are equal.

And, if the forces P and P_2 were acting alone their moment Pe would cause the resisting moment $\frac{s_2 I}{c}$ (Fig. 172*b*) such that

$$Pe = \frac{s_2 I}{c}.$$

The combined effect of the axial force and couple is to produce a maximum compressive unit-stress s such that

$$s = s_1 + s_2 = \frac{P}{a} + \frac{Mc}{I} = \frac{P}{a} + \frac{Pec}{I},$$

and the minimum stress is $\frac{P}{a} - \frac{Pec}{I}$. Or, since I may be expressed by the equation $I = ak^2$ in which k is the radius of gyration of the area a with respect to the centroidal axis, the unit-stress at any point whose distance from the centroidal axis is c may be found from the equation

$$s = \frac{P}{a}\left(1 \pm \frac{ec}{k^2}\right),$$

in which s must not exceed the proportional limit of the material.

Limitation on Eccentricity to Prevent Tensile Stress.—If the unit-stress s_2 at B (Fig. 172*b*) due to the moment Pe is greater than the unit-stress s_1 due to the axial load P_1, the resulting normal unit-stress at B will be a tensile stress. If, then, a tensile stress is to be avoided, as is usually desired in brittle material such as concrete, brick, cast iron, etc., which are relatively weak in tension, the value of e should not be greater than that found by making s_2 equal to s_1. Thus,

$$\frac{Pec}{I} = \frac{P}{a}, \quad \text{or} \quad \frac{Pec}{ak^2} = \frac{P}{a},$$

Hence

$$e = \frac{k^2}{c}.$$

Therefore, a short compression member will not be subjected to a tensile stress if the eccentric load acts on an axis of symmetry of the cross-section at a distance not greater than $\frac{k^2}{c}$ from the central axis of the member.

If the member has a rectangular cross-section the value $\frac{k^2}{c}$ is $\frac{1}{6}b$ or

$\frac{1}{6}h$ where b and h are the dimensions of the cross-section. This fact is expressed by the common rule that in the design of masonry structures the load should not lie outside the middle-third of the central axes of a rectangular cross-section. If the cross-sectional area of the member is circular the value of $\frac{k^2}{c}$ is $\frac{1}{8}d$, where d is the diameter, and hence the load should not lie outside the middle-fourth of any diameter of the section if tensile stress in the member is to be avoided.

ILLUSTRATIVE PROBLEM

Problem 270.—The machine member shown in Fig. 173(a) is acted on by a force P of 3000 lb. Find the maximum normal unit-stress on the area of the section AB at the wall. The dimensions of the cross-section of the bar at section AB is $\frac{3}{4}$ in. by 3 in.

Solution.—The load P may be resolved in two components, a cross-bending load P_y and a longitudinal eccentric load P_x (Fig. 173b); and P_x may be resolved further

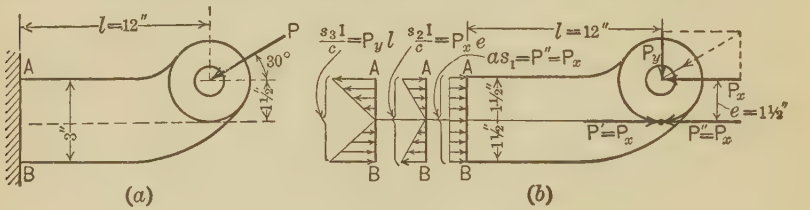

FIG. 173.

into an axial load P'' and a couple having a moment $P_x e$, by introducing the two equal opposite and collinear forces P' and P'', each equal to P_x, as indicated in Fig. 173(b).

Now since P'' is an axial load with respect to the section AB, it would, if acting alone, cause, and be held in equilibrium by, a total stress as_1, the unit-stress s_1 being constant over the area a of the section as indicated in Fig. 173(b). And the couple $P_x e$, if acting alone, would develop a resisting moment $\frac{s_2 I}{c}$ equal to the moment $(P_x e)$ of the couple as shown in Fig. 173(b). Likewise the cross-bending load P_y would develop a resisting moment $\frac{s_3 I}{c}$ equal to the moment $(P_y l)$. The shearing stress on the area is neglected in this discussion.

The normal unit-stress on the top fiber at A, then, is

$$s_A = -s_1 - s_2 + s_3$$

$$= -\frac{P\cos 30°}{2.25} - \frac{P\cos 30°\cdot 1.5\cdot 1.5}{\frac{1}{12}\frac{3}{4}\cdot(3)^3} + \frac{P\sin 30°\cdot 12\cdot 1.5}{\frac{1}{12}\frac{3}{4}\cdot(3)^3}$$

$$= -\frac{2600}{2.25} - \frac{2600\cdot 2.25}{1.69} + \frac{1500\cdot 18}{1.69}$$

$$= -1150 - 3460 + 16{,}000$$

$$= 11{,}390 \text{ lb. per sq. in., tensile stress.}$$

And at B the unit-stress is,

$$s_B = -s_1 + s_2 - s_3 = -1150 + 3460 - 16{,}000$$
$$= -13{,}690 \text{ lb. per sq. in., compressive stress.}$$

PROBLEMS

(In each of the following problems the load lies in a plane of symmetry of the cross-section of the area on which the stress is to be found.)

271. Solve Prob. 270, assuming P to act at the point where its action line intersects the longitudinal central axis of the member (the dotted line in Fig. 173a).

272. In Fig. 174 is shown a machine member having a rectangular cross-section 1 in. by 4 in. It is acted on by a force P_2 of 18,000 lb. and a force P_1, the action lines of which are shown in the figure. Find the value of P_1 if the maximum tensile unit-stress is 20,000 lb. per sq. in. *Ans.* $P_1 = 11{,}270$ lb.

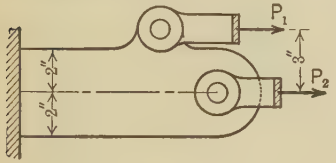

FIG. 174.

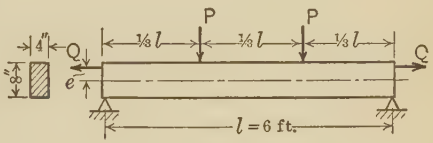

FIG. 175.

273. What value should e, in Fig. 175, have in order that the unit-stress at the top fiber within the middle third of the beam, due to the load Q, shall be equal (and opposite) to the unit-stress due to the cross-bending loads; assume $Q = 9P$.

274. The frame shown in Fig. 176 is used for small riveting, punching and stamping machines. Find the unit-stress developed at A and at B when the load P is 2000 lb.

Ans. $s_A = 1470$ lb. per sq. in., tension; $s_B = 1870$ lb. per sq. in., compression.

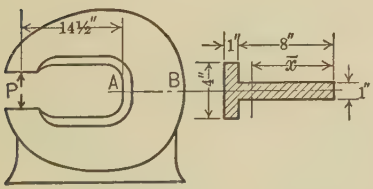

FIG. 176.

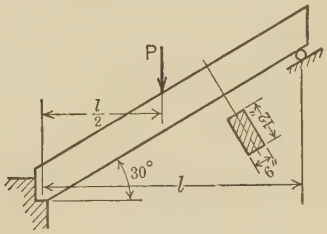

FIG. 177.

275. The inclined beam shown in Fig. 177 carries a load P of 5000 lb. at the mid-span. If the length, l, is 10 ft., what is the maximum compressive unit-stress developed in the beam? *Ans.* $s = 1050$ lb. per sq. in.

276. The post DE of the wall bracket shown in Fig. 171(a) is made of two 6-in. 15.5-lb. channel sections latticed together, the 6-in. dimensions being parallel to the direction of the boom BC. Find the maximum compressive unit-stress on a section of the post just beneath the section passing through A. Assume the load to be at the outer end of the boom BC.

277. A cast-iron machine frame shown in Fig. 178 is subjected to a load P of 8000 lb. The area of the cross-section at AB is 40 sq. in. and the centroidal axis YY is 6 in. from the outer edge of the section. The moment of inertia of the area with respect to the centroidal axis is 400 in.^{4} Find the maximum tensile and compressive unit-stresses on the section AB.

Ans. s_A = 1960 lb. per sq. in., tension; s_B = 2440 lb. per sq. in., compression.

278. The small crane shown in Fig. 179 has a clear swing of 28 in. Find the load P which will cause a maximum compressive unit-stress of 9000 lb. per sq. in. on the inner edge at B. *Ans.* P = 1060 lb.

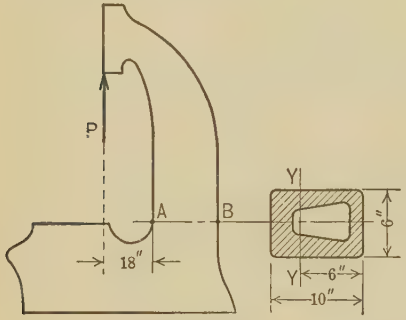

FIG. 178.

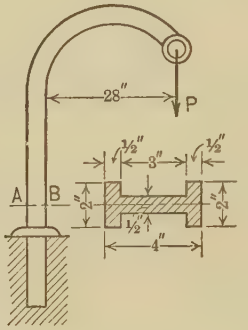

FIG. 179.

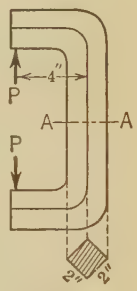

FIG. 180.

279. Calculate the maximum tensile unit-stress at section AA in the steel member shown in Fig. 180, assuming the load P to be 4000 lb. Show by a sketch the approximate stress distribution on section AA.

280. A compressive load P is applied parallel to the axis of a circular prism 12 in. in diameter, at a point 3 in. from the axis. If the maximum compressive stress developed is 1200 lb/in.^{2}, what is the value of P?

281. The open link shown in Fig. 181 is made of a steel bar having a diameter of 2 in. If the tensile unit-stress at A is 18,000 lb. per sq. in. what is the value of the load P?

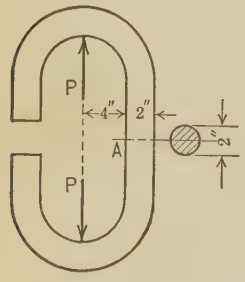

FIG. 181.

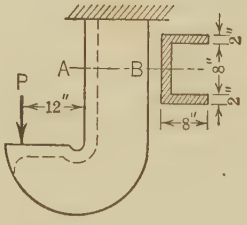

FIG. 182.

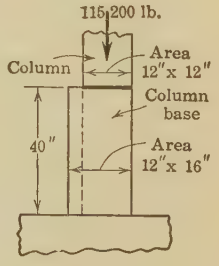

FIG. 183.

282. Calculate the tensile unit-stress at A in the cast-iron bracket shown in Fig. 182, due to a load P = 16,000 lb.

283. A column base (see Fig. 183) for a column 12 in. by 12 in. in cross-section was specified to be of the same cross-section as the column and 40 in. long. The contractor had no 12 in. × 12 in. column base but used one 12 in. × 16 in. as shown in Fig. 183. Show that the maximum calculated stress in a 12 in. × 12 in. base would be 800 lb. per sq. in., whereas in the 12 in. × 16 in. base it is 1050 lb. per sq. in. The safe or working stress for the timber was specified to be 800 lb. per sq. in. Would the column base be safer if the excess material were sawed off along the dotted line?

284. A cast iron T-section which has a flange 2 in. by 12 in. and a stem 2 in. by 12 in. is loaded as a short compression member. The resultant pressure on the end of the member acts through the center of the stem. If the load is 480,000 lb., what are the maximum and minimum unit-stresses produced by this eccentric load?

Ans. Max. = 28,000 lb./in.² comp. Min. = 1450 lb/in.² comp.

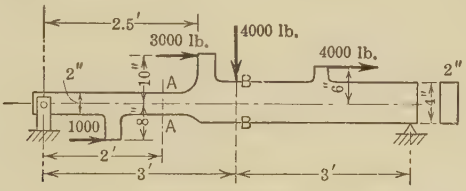

FIG. 184.

235. Calculate the maximum tensile (or compressive) unit-stress at sections AA and BB of the member shown in Fig. 184.

Ans. s_A = 19,300 lb./in.² tens.; s_B = 13,800 lb./in.² tens.

65. Eccentric Loads on Riveted Connections.—Examples of eccentric shearing loads are found frequently in riveted joints. Whenever feasible, however, eccentric loads should be avoided; that is, the action line of the resultant force to be transmitted through the rivets should pass through the centroid of the total rivet shearing area. Care in securing this condition, however, frequently is not given the attention it merits. The shearing stresses in riveted joints subjected to eccentric loads may be found as follows.

Let it be required to find the shearing unit-stress in each rivet in the riveted joint shown in Fig. 185(*a*). A force P having the eccentricity e is transmitted from the member A to the gusset plate B, and from the gusset plate through the four rivets to the member C. The shearing area of each rivet will be denoted by a, the diameter by d, and the distances of the centers of rivet areas from the centroid, G, of the total shearing area by l_1, l_2, etc.

The load P may be resolved into a load P_1 (Fig. 185*b*) acting through the centroid of the total shearing area and a twisting or torsional couple P, P_2 having the moment Pe. The central shearing load P_1 if acting alone would develop the same shearing unit-stress, s'_s, on each of the rivet areas (or rather this is the assumption usually made, see Art. 17).

The resisting stress that holds P_1 in equilibrium, then, is $4as'_s$. Thus,

$$P_1 = 4as'_s \quad \text{or} \quad s'_s = \frac{P}{4a},$$

as indicated in Fig. 185(*b*).

And, if the couple Pe were acting alone it would develop a resisting moment $\left(\frac{s''_s J}{c}, \text{ Art. } 21\right)$,[4] the shearing unit-stress in the rivets varying

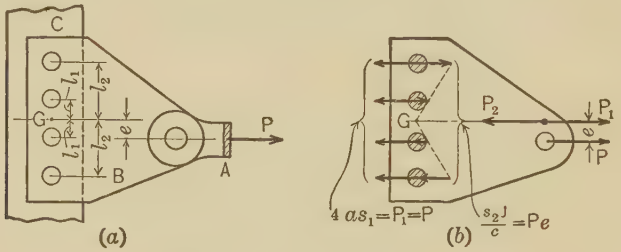

FIG. 185.—Shearing stress in rivets due to eccentric load.

directly as the distances of the rivets from the centroid G, as shown in Fig. 185(*b*). This resisting moment holds the moment Pe in equilibrium. Hence,

$$Pe = \frac{s''_s J}{c} \quad \text{or} \quad s''_s = \frac{Pec}{J}.$$

The shearing unit-stress on the bottom rivet, then, is

$$s_s = s'_s + s''_s$$

$$= \frac{P}{4a} + \frac{Pec}{J}, \quad \ldots\ldots\ldots \quad (118)$$

in which c is the distance from G to the rivet on which the stress is desired (equal to l_2 for the bottom rivet), and J is the polar moment of inertia of the total shearing area of the rivets with respect to an axis passing through G.

Since the diameters of the rivet areas are small in comparison with the distances of the rivets from G, the value of J is, with slight error,

[4] In Art. 21 it was stated that $\frac{s_s J}{c}$ is the expression for the resisting moment only for a solid or hollow cylindrical shaft. The rivet areas, however, may be transformed into annular areas having radii l_1 and l_2, similar to the cross-section of two concentric hollow cylindrical shafts.

the sum of the products obtained by multiplying each rivet area by the square of the distance of its center from G. Thus,

$$J = 2a(l_1^2 + l_2^2)$$

and hence

$$Pe = \frac{2as''_s}{l_2}(l_1^2 + l_2^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

Instead of obtaining the resisting moment from the expression $\frac{s''_s J}{c}$, it may also be found as the sum of the moments, with respect to G, of the total shearing stresses on the rivet areas. Thus if s''_s denotes the shearing unit-stress on the rivets farthest from G, and s''_{s1} on the rivets nearest to G, we have

$$\begin{aligned} Pe &= 2[as''_s l_2 + as''_{s1} l_1] \\ &= 2\left[as''_s l_2 + a\left(\frac{l_1}{l_2}s''_s\right)l_1\right], \text{ since } s''_{s1} = \frac{l_1}{l_2}s''_s \\ &= 2\frac{as''_s}{l_2}(l_1^2 + l_2^2), \end{aligned}$$

which is the same as equation (119).

PROBLEMS

286. A building was constructed with a number of riveted connections like that shown in Fig. 186. The cross-sectional area of each rivet is 0.40 sq. in. Show that the maximum shearing stress in the rivets of this joint (at A) is 12,000 lb. per sq. in. Show also that if the rivet at D is omitted the shearing unit-stress in each of the remaining rivets is 10,000 lb. per sq. in. The specifications called for a working shearing stress of 10,000 lb. per sq. in. The contractor found it more convenient to use a larger gusset plate than was specified and added the extra rivet at D, thinking he was making a stronger joint than was specified. Would the connection be improved by requiring the contractor to knock out the rivet at D?

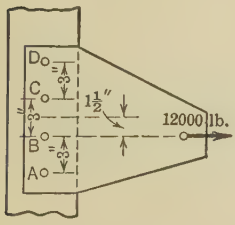

FIG. 186.

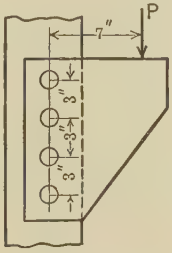

FIG. 187.

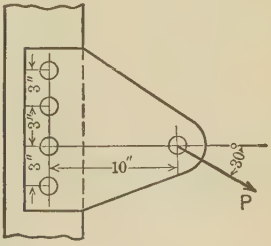

FIG. 188.

287. The diameter of the rivets in the joint shown in Fig. 187 is $\frac{3}{4}$ in., and the load P is 4000 lb. Find the shearing unit-stress in the top and bottom rivets.

Ans. s = 6740 lb. per sq. in.

288. The load P transmitted through the joint shown in Fig. 188 is 5000 lb. and

the diameter of the rivets is $\frac{3}{4}$ in. Find the shearing unit-stress in the top and in the bottom rivet.

66. Helical Spring.—*Stress Developed.*—A helical or coil spring (Fig. 189*a*) is frequently used to resist an axial load P that lengthens or shortens the spring. The stress developed in the rod of which the spring is made is mainly torsional shearing stress. The value of the shearing unit-stress in a spring made of rod having a *circular cross-section* may be found as follows:

Let a section be passed through the rod at A (Fig. 189*a*) and the lower part of the spring be removed. The part above the section A is shown in Fig. 189(*b*). Further, let the load P be resolved into a force P_1 (whose action line passes through the center of the section at A) and

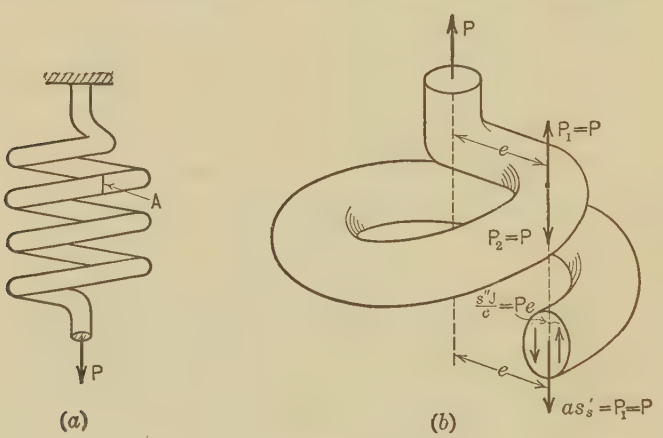

FIG. 189.—Stress in helical spring.

a twisting moment Pe, by introducing the two forces P_1 and P_2 (Fig. 189*b*) each equal to P. Now these forces (which produce the same stress on the section at A as does the original one force P) are held in equilibrium by the stresses on the section at A. If the force P_1 were acting alone it would produce a resisting shearing stress that would hold P_1 in equilibrium, and this direct shearing stress would be equal to as'_s as indicated in Fig. 189(*b*), a being the area of cross-section of the rod, and s'_s the shearing unit-stress. Hence,

$$P_1 = P = as'_s \quad \text{or} \quad s'_s = \frac{P}{a}.$$

Further, if the twisting moment Pe due to the couple P, P_2 were acting alone it would produce a resisting moment $\frac{s''_s J}{c}$ on the area a which

would hold the external or twisting moment in equilibrium as indicated in Fig. 189(*b*). Hence

$$Pe = \frac{s''_s J}{c}.$$

Now the direct shearing unit-stress s'_s in most springs is negligible in comparison with the torsional unit-stress s''_s, and hence the chief unit-stress in the rod is due to torsion and its maximum value is

$$s''_s = \frac{Pec}{J},$$

in which e is the mean radius of the coil, c the radius of the rod (equal to $\frac{d}{2}$ where d is the diameter of the rod) and J is the polar moment of inertia of the cross-sectional area of the rod $\left(J = \frac{\pi d^4}{32}, \text{Art. } 171\right)$.

The above expression then may be written

$$s''_s = \frac{\frac{Ped}{2}}{\frac{\pi d^4}{32}} = \frac{16Pe}{\pi d^3}.$$

If, however, the direct unit-stress s'_s is not negligible, the maximum unit-stress on the area a will be the sum of s'_s and s''_s. Thus

$$s_s = \frac{P}{a} + \frac{16Pe}{\pi d^3} = \frac{P}{a}\left(1 + \frac{4e}{d}\right) \quad . \quad . \quad . \quad . \quad . \quad (120)$$

Deflection of Helical Spring.—The increase or decrease in length (called deflection) of the spring due to a given load P may be found as follows:

If the shearing proportional limit of the material is not exceeded, the work done by the load, as it increases uniformly from zero to the value P, is stored in the material as stress (potential) energy, and further, if the stress developed is assumed to be due only to torsion, it follows that

Work done in causing deflection = Work done in twisting the rod.

Now the work done by the load in deflecting the spring is the product of the average value of the load $\left(\frac{P}{2}\right)$ and the deflection Δ. Similarly,

the work done in twisting the rod through an angle θ is the product of the average value of the twisting moment $\left(\frac{T}{2} \text{ or } \frac{Pe}{2}\right)$ and the angle of twist. Hence,

$$\tfrac{1}{2}P\Delta = \tfrac{1}{2}T\theta = \tfrac{1}{2}Pe\theta.$$

Therefore,

$$\Delta = e\theta$$

But from Art. 24, the value of θ is given by the equation,

$$\theta = \frac{Pel}{E_s J}.$$

Further, for a closely coiled spring, $l = 2\pi en$, approximately, where n is the number of turns. Hence,

$$\Delta = e\frac{Pe2\pi en}{E_s J}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

but, since $J = \frac{\pi d^4}{32}$, the above expression becomes,

$$\Delta = \frac{64Pe^3 n}{E_s d^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

If Δ is expressed in inches and P in pounds, then e and d are expressed in inches, and E_s in pounds per square inch. If the shearing unit-stress developed in the bar is greater than the shearing proportional limit the above equation is not valid.

The deflection, Δ_s, due to the direct shear is $\Delta_s = \frac{Pl}{aE_s}$ and hence the total deflection is

$$\text{total deflection} = \frac{Pl}{aE_s} + \frac{Pe^2 l}{E_s J} = \frac{Pl}{aE_s}\left(1 + \frac{e^2}{k^2}\right), \quad . \quad . \quad (123)$$

in which k is the polar radius of gyration of the section.

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289. A helical spring when compressed 4 in. must support an axial load of 1600 lb. The diameter of the rod is $\frac{5}{8}$ in., the mean diameter of the coil is 3 in. and the working shearing unit-stress of the material (spring steel), is 60,000 lb. per sq. in. Find the number of coils required. Is the working stress exceeded?

Ans. $n = 21.2$ coils. $s_s = 55{,}200$ lb. per sq. in.

290. A helical spring made of steel rod $\frac{1}{4}$ in. in diameter with 5 turns is 2.65 in. high and has an outside diameter of 1.35 in. Find the modulus of the spring, that is, the force required to stretch it an inch, assuming that the proportional limit of the material is not exceeded.

Ans. Mod. = 880 lb., if deflection due to direct shear is neglected.

291. A bronze helical spring is required to support 980 lb. when deflected 3 in. Find a suitable diameter of the coil and number of turns if the diameter of the rod is $\frac{1}{2}$ in. and the working unit-stress is not to exceed 50,000 lb. per sq. in. $E_s = 6{,}000{,}000$ lb. per sq. in.

REVIEW QUESTIONS AND PROBLEMS

292. What is the principle of superposition? Explain why it does not apply to stresses above the yield point of a material?

293. Derive and interpret the equation $s = \frac{P}{a} \pm \frac{Pec}{I}$. Explain why this equation is not applicable to a member if the unit-stress in the member exceeds the proportional limit of the material.

294. An eccentric load is applied to the end of a compression block having a rectangular cross-section; the action line of the load is parallel to the geometric axis of the block, and the maximum unit-stress developed in the block is within the proportional limit of the material. What is accomplished by restricting the point of application of the load to a point within the middle third of an axis of symmetry of the area? Draw a sketch to explain your answer, showing the stress distribution on a section of the block when the load acts at one end of the middle third length.

295. The cast-iron frame shown in Fig. 190 is subjected to a load P of 24,000 lb. and two loads P_1, each equal to 1000 lb. Find the maximum tensile and compressive unit-stresses in the frame. What are the factors of safety for these stresses based on the ultimate strengths of the material? (See Table 1 in Chapter I.)

Ans. $s_t = 5940$ lb. per sq. in.; $s_c = 7240$ lb. per sq. in.; $f_t = 3.4$; $f_c = 10.4$.

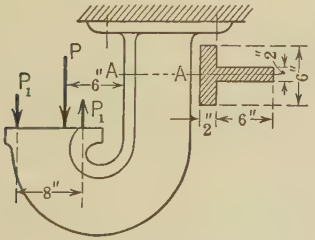

FIG. 190.

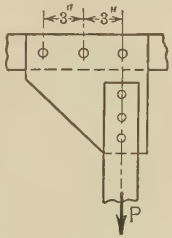

FIG. 191.

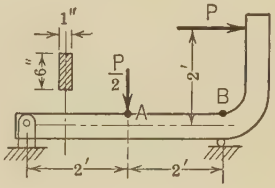

FIG. 192.

296. Find the maximum shearing unit-stress developed in the rivets in the riveted connection shown in Fig. 191 if the load P is 8000 lb., and indicate the rivet in which this stress is developed. Each rivet is $\frac{7}{8}$ in. in diameter.

Ans. $s_s = 11{,}100$ lb. per sq. in.

297. If the maximum working unit-stress for the member shown in Fig. 192 is 16,000 lb. per sq. in., what is the maximum value of the load P that can be applied to the member? Where in the member does the maximum stress of 16,000 lb. per sq. in. occur? *Ans.* $P = 3840$ lb.; at B.

CHAPTER VIII

COMPRESSION MEMBERS. COLUMNS

67. Introduction.—A compression member is a bar (either solid or built-up, and, as a rule, approximately straight) subjected to an end-load (or loads) acting parallel to the axis of the bar. If the load acts through the centroid of the end section it is called an axial load; if not, it is called an eccentric load. But, an eccentric load is equivalent to an axial load and an end-moment or couple. The character of end-loads applied to compression members as used in engineering structures and machines depends largely on the type of end-connection used in the structure, and on the relative stiffness of the members connected. For example, a locomotive connecting rod, many members in certain types of bridges, etc., are connected by pins, and the end-load exerted on a compression member by the pin is mainly an axial force, since the end-moment due to the friction of the pin is relatively small; whereas, most members in buildings, etc., are connected by being welded or riveted either directly or by means of an auxiliary piece such as a gusset plate, and a compression member may be subjected to an end-moment, in addition to a compressive load, due to the bending of a member (a beam, say) to which the compression member is attached.

If a compression member is slender (relatively long), it is usually called a column; and if relatively short it is called a compression block or a strut. This chapter deals mainly with the strength of columns; there is, however, no definite slenderness that differentiates a column from a compression block.

In the investigation of the stresses and deformations in a column there must be considered, as affecting the action of the column, certain conditions that may be neglected in the investigation of the stresses and deformations in most other structural members. For example, in determining the stresses and deformations in beams, tension members, torsional shafts, etc., when resisting loads, the member in question is considered to satisfy ideal conditions; namely, the body is considered to be straight, to be made of homogeneous material free from initial stresses, to be so constructed that it acts as a whole or as a unit, and to

be subjected to loads having definitely known action lines. And, although it is known that such ideal conditions are never fully realized in these structural members, experiments and experience have shown that, in general, the results found from an analysis of the actions in the ideal beam, tension member, etc., are reliable; that is, the primary or significant action in the actual member is the same as that in the ideal body. But, the action in an actual column, as shown by experiments and experience, may deviate much from that in an ideal column. This fact may be stated in a somewhat different way as follows: In beams, tension members, etc., conditions that differ from those in the ideal members do not have more than a secondary effect on the stresses and deformations in the beams, tension members, etc., whereas the similar conditions in a column may be of prime importance in determining the resistance of the column.

Further, a quantitative measure of the effects of these conditions cannot, in general, be determined, and hence they render the analysis of column action less definite and satisfactory than that of beams, tension members, etc., and at the same time, cause a column to be a more critical member in a structure than most other structural members.

A brief discussion of the actions and conditions that must be especially considered in a column are given in the next article and also in Art. 74.

68. Distinguishing Features of Column Action.—The fundamental idea of column action is that of combined compression and bending, but the bending action in a column is accompanied by certain conditions that renders a quantitative measure of the bending action less certain than that of the bending in a beam or in a member subjected to combined tension and bending, etc. The distinguishing features of column action may be stated briefly as follows:

1. A straight column may bend or deflect as a whole when subjected to a supposedly axial end-load (Fig. 193*a*), the initial bending being due to a slight initial or accidental eccentricity [1] of loading, and this deflection causes an increase in the moment-arm of the load, and hence increases the bending moment; the increased bending moment in turn increases the deflection, etc., tending to cause failure of the column by flexure or what is frequently called "buckling." This action is of special importance in the more slender columns and in the component parts of built-up columns.

[1] If the column bends there must be a bending moment to start the bending; various conditions contributing to this bending are discussed later but an initial or accidental eccentricity may here be considered to be the chief cause of the initial bending.

In a column, then, the bending moment may increase more rapidly than the load increases, whereas this undesirable condition does not occur in other structural members. For example, in a straight compression block the deflection may be assumed to be negligible and hence the initial bending moment, if any, due to initial eccentricity, etc., increases only as the load increases since the moment-arm of the load remains constant. Likewise, in the case of a beam, although the deflection of the beam may be greater than that of a column, the deflection has a negligible effect on the moment-arm of the load. Again, in the case of a tension member subjected to an eccentric load, the bending of the member causes the moment-arm of the load to decrease, and hence the bending moment increases even less rapidly than the load increases.

2. By virtue of the action discussed under (1) above, an initial crookedness in a column as a whole (and all columns are more or less crooked) reduces the strength of a column more than a like crookedness reduces the strength of a beam, tension member, etc.

3. Local kinks or bends in the component parts of a built-up column, lack of homogeneity of material, initial stresses in the material, and poorly designed end-connections, may cause stresses and deformations in parts of a column greatly in excess of those assumed to exist in the corresponding ideal column, and these localized stresses and deformations, as a rule, reduce the resistance of the column more than similar localized stresses reduce the resistance of beams, tension member, etc.

4. A variation in the position of the action line of a load on a column from that assumed in the ideal column, that is, the presence of accidental or fortuitous eccentricities, cause, in general, a much greater increase in the stresses in a column than similar variations cause in the stresses in beams, in tension members, etc. Such fortuitous eccentricities are very likely to occur due to the actions of the end-connections.

On account of the indefiniteness of the factors affecting the strength of columns, the symbol representing unit-stress as used in the column formulas that are discussed in the following articles should not be thought of as the value of the actual unit-stress in the column caused by a given load as was done (in Chap. IV) in connection with beams, but should be thought of merely as a nominal value (in the same units as unit-stress) which cause the formulas to give the value of the load the column should support under the conditions stated.

69. Slenderness Ratio.—The slenderness ratio for a column is a term that enters in all column formulas; it is the ratio of the length, l, of the column to the radius of gyration, r, of the cross-section with respect to the centroidal axis that is perpendicular to the plane in which the column

bends or tends to bend. If the column is pivot-ended, that is, free to turn at its ends, it will bend in a plane perpendicular to the axis of the cross-section about which the moment of inertia is least, and r will be the *least* radius of gyration. Column sections are frequently designed so that the moments of inertia of the section about the principal axes are approximately equal; this condition, however, is not always desirable.

70. Two Limiting Cases of Compression Members.—If a short solid compression block is subjected to an end-load having a very slight eccentricity, the block may be assumed to be in pure compression since the bending action may be neglected. But if this block be assumed gradually to become longer (more slender) the tendency to bend becomes greater, and in a steel member the bending cannot be considered to be negligible if the length of the member is greater than about ten times the least lateral dimension; that is, if the slenderness ratio is about 40. And, if the column is very slender, the resistance of the column is mainly its resistance to bending, the effect of the direct compression then being negligible. The slenderness ratios of compression members used in most engineering structures and machines, however, are such that both direct compression and bending are of importance, and such members will be discussed after considering the two limiting cases of compression members.

One limiting case of a compression member, then, is a compression block, and for such a member, when made of homogeneous material and subjected to an axial load P, the formula is

$$P = as,$$

in which a is the cross-sectional area of the member and s is the unit-stress at all points in the area.

The other limiting case is that of a member so slender that bending or flexure is the only action that need be considered, and hence the flexure formula, $M = \frac{sI}{c}$, for such a column applies, the bending moment M at any section being Py. (Fig. 193*a*.) Thus,

$$Py = \frac{sI}{c},$$

which states that the bending moment (Py) of the external forces about any section is held in equilibrium by the resisting moment $\left(\frac{sI}{c}\right)$ of the

stresses developed on that section. But the value of the deflection y in the equation corresponding to a given load P is unknown and hence the value of P that produces a given stress s cannot be found, since there are two unknown quantities in the equation; in other words, the problem is statically indeterminate; but, as discussed in Chapter VI, the elastic curve equation offers another method for finding external forces. The expression for the load required to hold a slender column in a bent position as found from the elastic curve of the column was first obtained by Euler in 1757 and is known as Euler's column formula; it is derived in the following article.

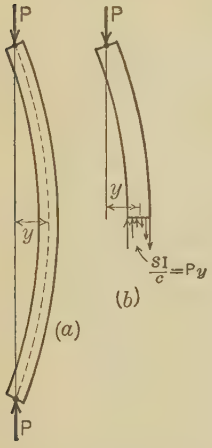

Fig. 193.—Bending of slender column.

71. Euler's Column Formula.—The problem is to find the *least value of an axial load P that will cause a column to bend.* The importance of this value of the load lies in the fact that, owing to the conditions stated in Art. 68, a column fails by sudden bending (called buckling) if the load is increased only a small amount above this value, and hence the least value of the load that causes a column to bend *is also the maximum load the column can resist,* provided that the column is so slender that the direct compressive stress is negligible and hence the column does not fail by crushing of the material but fails by buckling.

Now the method of obtaining this critical bending load is to assume that an ideal column is deflected by a lateral force [2] and that an axial load P is applied that will maintain this deflection after the lateral force is removed; a relation between the load and the deflection is then found from the elastic curve equation of the deflected column, and the limiting value of P found from the equation, by assuming that the deflection becomes indefinitely small, is the least value of P that will cause an ideal column to bend (or the maximum value that will not cause bending).

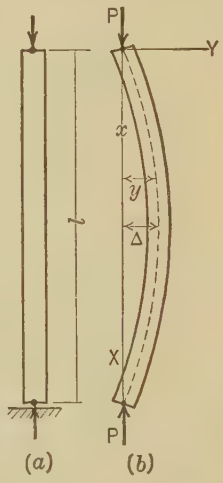

Fig. 194. — Elastic curve of slender column.

In Fig. 194(*a*) is represented a slender solid column of constant cross-section; it is assumed to be straight, to be made of homogeneous material free from initial stresses, and to be

[2] An axial load could not cause an *ideal* column to bend.

subjected to an axial end-load only, that is, no restraints occur at the ends and the weight of the column is neglected. Let the load P be such as to hold the column in the deflected position as shown in Fig. 194(b). Now since bending is the only action here considered the equation of the elastic curve is the same as that of a beam; namely (see Art. 43),

$$EI\frac{d^2y}{dx^2} = M,$$

in which M is equal to Py. If the origin of axes is chosen at the upper end of the column and the positive direction of the x axis is downward, and the positive direction of the y axis is the direction of the deflection, as shown in Fig. 194(b), then $\frac{d^2y}{dx^2}$ is negative and the above equation becomes

$$EI\frac{d^2y}{dx^2} = -Py. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

This equation may be written

$$EI\frac{d\left(\frac{dy}{dx}\right)}{dx} = -Py, \quad . \quad . \quad . \quad . \quad . \quad . \quad (124a)$$

and if each side of the equation is multiplied by dy, each side becomes an exact derivative of the form udu; namely,

$$EI\left(\frac{dy}{dx}\right)d\left(\frac{dy}{dx}\right) = -Pydy.$$

Integrating, we have,

$$EI\left(\frac{dy}{dx}\right)^2 = -Py^2 + C_1. \quad . \quad . \quad . \quad . \quad . \quad (125)$$

Now simultaneous values of $\frac{dy}{dx}$ and y are not known, and hence C_1 cannot be expressed in terms of the known quantities E, I and l, but it may be expressed in terms of the maximum deflection, Δ. Thus, since $\frac{dy}{dx} = 0$ when $y = \Delta$, we have

$$0 = -P\Delta^2 + C_1; \quad \text{whence} \quad C_1 = P\Delta^2,$$

and eq. (125) becomes

$$EI\left(\frac{dy}{dx}\right)^2 = P(\Delta^2 - y^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

This equation may be integrated after the variables are separated; the equation may be written as follows:

$$\frac{dy}{\sqrt{\Delta^2 - y^2}} = \sqrt{\frac{P}{EI}}\,dx.$$

Integrating, we have

$$\sin^{-1}\frac{y}{\Delta} = \sqrt{\frac{P}{EI}}\,x + C_2, \quad \ldots \quad (127)$$

or

$$\frac{y}{\Delta} = \sin\left(\sqrt{\frac{P}{EI}}\,x + C_2\right).$$

But $y = 0$ when $x = 0$ and hence $C_2 = 0$. Therefore,

$$\frac{y}{\Delta} = \sin\sqrt{\frac{P}{EI}}\,x \quad \text{or} \quad y = \Delta \sin\sqrt{\frac{P}{EI}}\,x. \quad \ldots \quad (128)$$

Now if the deflection of a column is considered to decrease, the value of P required to maintain the deflection also decreases and attains its limiting (minimum) value when the deflection becomes indefinitely small. But, the column is then virtually straight, and the vertical distance between the ends of the column may be assumed to be equal to l. Therefore, the limiting value of P may be found by substituting the values $y = 0$ when $x = l$ in eq. (128). Whence,

$$\sin\left(\sqrt{\frac{P}{EI}}\cdot l\right) = 0,$$

and hence the angle $\left(\sqrt{\frac{P}{EI}}\cdot l\right)$ must be equal to π or some integral multiple of π; the value is π for a column with pivoted ends and 2π for one with fixed ends, etc. (see Art. 73). Therefore, the minimum value of P for a column having pivoted ends is

$$P = \frac{\pi^2 EI}{l^2}. \quad \ldots \quad (129)$$

An axial load P, then, less than that given by eq. (129) will not cause a column to bend, whereas a load greater than this will cause an *actual* column to bend, or will maintain an *ideal* column in a deflected position. And, although an ideal slender column may resist a somewhat

larger load than this, the accompanying deflection becomes excessive[3] when P has a value only slightly greater than that given by eq. (129). Therefore, the value of P in eq. (129) is considered to be the maximum load an actual (structural) slender column can resist without *failing by sudden bending*. A less slender column may fail, however, by crushing or yielding of the material by direct compression before this Euler critical value is reached, as will be discussed later.

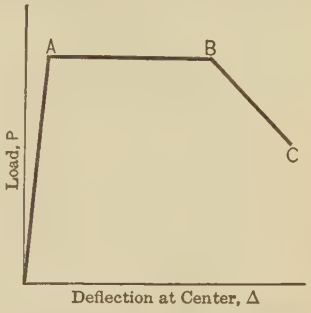

FIG. 195.—Relation between load and deflection for slender column.

The fact that excessive deflections occur when P has a value close to the Euler load is clearly shown by testing a slender column, such as a wooden airplane strut, and measuring the deflection. The load-deflection curve obtained is shown in Fig. 195; the deflection starts somewhat before the Euler load is reached due to slight eccentricities, etc., but at the Euler load the deflection increases rapidly to a value equal to AB with very slight increase of load, the stress in the column being less than the proportional limit until the deflection AB is exceeded.

Since I in eq. (129) is equal to ar^2 (Art. 169) where a is the area of cross-section and r is the radius of gyration of the cross-section about the neutral axis, Euler's equation may be written

$$\frac{P}{a} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}, \quad \ldots \ldots \ldots \quad (130)$$

in which $\frac{l}{r}$ is the slenderness ratio of the column.

Eq. (130) shows that the only property of the material on which P depends is the stiffness of the material, E; therefore, a slender column made of high-carbon steel or special alloy steel would begin to bend, and hence fail, at the same load as would one made of low-carbon steel,

[3] This fact is not evident from the analysis of the column action here presented; it is shown by the true equation of the elastic curve, that is, the equation obtained when the assumption that the length along the axis is the same as the length along the curve ($dx = dl$) is not made. The truth of the statement, however, is clearly shown by experiments as indicated by the results represented in Fig. 195.

provided that the $\frac{l}{r}$ were relatively large so that the columns would not fail by crushing before the Euler load is reached.

72. Graphical Representation of Formulas for Ideal Columns.—As already noted, the maximum axial load P that an ideal column having a very small slenderness ratio (a compression block) can resist depends on the compressive strength s of the material and the cross-sectional area a, the equation being $\frac{P}{a} = s$, whereas the maximum load that a slender column can resist depends on the stiffness, E, (not strength) of the material and on the slenderness ratio $\frac{l}{r}$, in addition to the area a; the equation being $\frac{P}{a} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$.

Now the results of tests of axially loaded columns agree approximately with the calculated values from the above equations when the columns have small and large values, respectively, of $\frac{l}{r}$. Thus, if a series of ideal columns were made of one kind of material but with different slenderness ratios, and were tested to failure by applying axial loads, the columns having small values of $\frac{l}{r}$ (less than about 40 in the case of steel) would fail by direct compression when the compressive unit-stress in the material reached the ultimate compressive strength of the material; the maximum usable compressive strength of ductile material in columns, however, is the yield-point of the material since at this stress plastic flow of the material occurs and the column fails to perform its structural function even if total collapse does not result. But the columns having large values of $\frac{l}{r}$ would fail by buckling when the unit-load $\frac{P}{a}$ is slightly greater than that given by Euler's formula; the unit-stress in the column when the buckling starts being less than the proportional limit of the material.

If, then, values of $\frac{P}{a}$ that cause failure are plotted (Fig. 196) as ordinates and the corresponding values of $\frac{l}{r}$ as abscissae the column

formula $\frac{P}{a} = s$ is represented by the horizontal line AB, and Euler's equation is represented by the curve CBD.

An ideal column having an $\frac{l}{r}$ greater than that represented by the abscissa OB' (Fig. 196) fails by bending, and an ideal column having a slenderness ratio less than this critical value fails by the yielding of the material in direct compression if the material is ductile, or by crushing or diagonal shearing of the material if the material is brittle.

The graph representing the column formulas for ideal columns is, therefore, the curve ABD of Fig. 196; the part BC of the Euler curve has no physical meaning, since the column would fail by crushing before a load represented by an ordinate to BC could be applied.

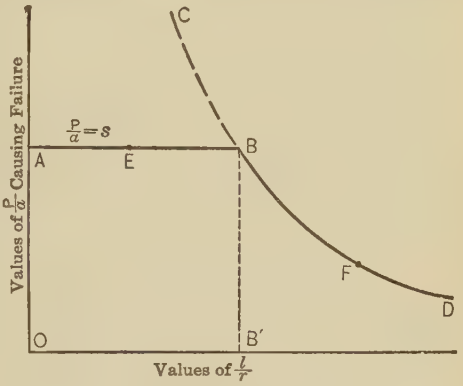

FIG. 196.—Graphical representation of ideal-column formulas.

ILLUSTRATIVE PROBLEMS

Problem 298.—Find the critical value of $\frac{l}{r}$, corresponding to OB' in Fig. 196, for an axially-loaded pivot-ended ideal structural-steel column.

Solution.—The yield point of structural steel is about 40,000 lb. per sq. in. and $E = 30{,}000{,}000$ lb. per sq. in. Now $\frac{P}{a}$ as found from the two column formulas are equal when $\frac{l}{r}$ has the value OB' (Fig. 196), and hence,

$$\frac{P}{a} = s = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$$

That is,

$$40{,}000 = \frac{\pi^2 \times 30{,}000{,}000}{\left(\frac{l}{r}\right)^2}, \qquad \text{whence,} \quad \frac{l}{r} = 86.$$

Therefore, an ideal axially-loaded pivot-ended structural-steel column would fail by

buckling if its slenderness ratio were greater than about 86. Tests of actual pin-ended steel columns, however, show that Euler's equation is applicable only when $\frac{l}{r}$ is greater than about 120, for reasons discussed in Art. 68 and in the following articles.

Problem 299.—A column made of a 10-in. 35-lb. I-beam and two plates $\frac{5}{8}$ in. by 11 in., as shown in Fig. 197, is 32 ft. long and is subjected, in a testing machine, to an axial load applied through spherical-seated bearings. Calculate the least radius of gyration, r, and the corresponding slenderness ratio, $\frac{l}{r}$. Find the maximum load the column can resist, and also a working load using a reduction factor (so-called factor of safety) of 3.

Solution.—The moment of inertia of the cross-sectional area about the x-axis may be found as follows: Area of each plate, $a_1 = 6.87$ sq. in. A steel maker's handbook gives:

Area of the I-section = 10.29 sq. in.
I_x of the I-section = 146.4 in.^{4}
I_y of the I-section = 8.5 in.^{4}

Hence,

Total area of cross-section, a = 24.03 sq. in.
I_x of the two plates $= 2(\bar{I} + a_1 d^2)$, (see Art. 170)
$= 2a_1 d^2$ approximately
$= 2 \cdot 6.87 \cdot (5.3)^2$
$= 386$ in.^{4}

FIG. 197.

I_x for whole area $= 146.4 + 386 = 532.4$ in.^{4}
I_y for whole area $= 8.5 + 2(\frac{1}{12} bh^3)$
$= 8.5 + 2(\frac{1}{12} \frac{5}{8})\ (11)^3$
$= 8.5 + 138.6 = 147.1$ in.^{4}

Since I_y is smaller than I_x the column will bend so that the y-axis is the neutral axis, that is, in a plane perpendicular to the y-axis.

The least radius of gyration, $r = \sqrt{\frac{I_y}{a}} = \sqrt{\frac{147.1}{24.03}} = 2.47$ in. Therefore,

$$\frac{l}{r} = \frac{32 \times 12}{2.47} = 155.5,$$

and this value is sufficiently large, as discussed in the next article, to allow the column to bend before failing by direct compression.

Hence,

$$\text{Max. unit-load} = \frac{P}{a} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{9.87 \times 30{,}000{,}000}{(155.5)^2} = 12{,}250 \text{ lb. per sq. in.}$$

$$P = 12{,}250 \times 24.03 = 294{,}200 \text{ lb.}$$

$$\text{Working load} = \frac{294{,}200}{3} = 98{,}100 \text{ lb.}$$

PROBLEMS

300. Find the least value of $\frac{l}{r}$ for an ideal pivot-ended oak column that will fail by bending when subjected to an axial load. Assume the following values for the ultimate compressive strength s_u, and modulus of elasticity: s_u = 8000 lb. per sq. in., E = 1,500,000 lb. per sq. in.

301. Solve Prob. 299, assuming that the column is made of rolled aluminum (consult Table 1 of Chapter I).

302. A 3-in. by 3-in. by $\frac{1}{2}$-in. wrought iron angle 15 ft. long was tested by the Pencoyd Steel Co., spherical seated bearings at the ends being used. It failed when subjected to an axial unit-load of 2650 lb. per sq. in. What is the $\frac{l}{r}$ for the column? Calculate the maximum unit-load the column would be expected to resist. (E = 25,000,000 lb. per sq. in.) *Ans.* $\frac{l}{r} = 310; \frac{P}{a}$ = 2570 lb. per sq. in.

303. Design by Euler's formula, a square pivot-ended timber column 24 ft. long to resist an axial load of 18,000 lb. Use a working load equal to 8 times the given load; assume E = 1,500,000 lb. per sq. in. Can the use of Euler's formula be justified in this problem? *Ans.* b = 9.92 in.

304. What is the ratio of the strength of an ideal slender solid cast-iron column 6 in. in diameter to the strength of a slender hollow cast-iron column having a wall thickness of 1 in., the areas of cross-section and the lengths of the two columns being equal? *Ans.* 9 to 41.

305. A spruce airplane strut is 66 in. long and has an elliptical cross-section with major and minor diameters of 4 in. and $1\frac{1}{2}$ in. respectively. Calculate the slenderness ratio of the column, and if Euler's formula is applicable, calculate the safe axial load this column will stand as a pivot-ended column using a factor of safety of 3. Assume E = 1,600,000 lb./in.^{2}, I about the neutral axis is $\frac{\pi bd^3}{64}$ = 0.663 in.^{4} and $a = \frac{\pi bd}{4}$ = 4.71 in.^{2} *Ans.* $\frac{l}{r}$ = 176.

306. Select a structural steel I section to serve as a column 20 ft. long to resist an axial load of 60,000 lb. Use a factor of safety of 4 and assume the column to have round or pivoted ends. Justify the use of Euler's formula by showing that the column is slender and that $\frac{P}{a}$ is less than the proportional limit of the material.

73. Effect of End-conditions.—I. ON IDEAL SLENDER COLUMNS.—The more important ideal end-conditions that are approached more or less closely in the end-connections of structural columns are illustrated in Fig. 198, and may be described as follows:

(*a*) Both ends free to turn but not free to move laterally. Fig. 198(*a*); a column subjected to such end-conditions is usually called a pivot-, round-, or hinge-ended column.

(*b*) Both ends fixed so that the tangents to the elastic curve at the ends are parallel to the original axis of the column, Fig. 198(*b*).

(*c*) One end fixed and one end free to turn but not free to move laterally, Fig. 198(*c*).

(*d*) One end fixed and one end free from all restraint, Fig. 198(*d*).

The axial load P that will cause the column shown in Fig. 198(*a*) to bend (considered to be the maximum load a slender column can resist, see Art. 71) is $P = \dfrac{\pi^2 EI}{l^2}$.

In Fig. 198(*b*) the points of inflection, A and C, are at a distance of $\frac{1}{4}l$ from the ends, and hence the middle half, ABC, is a column of the

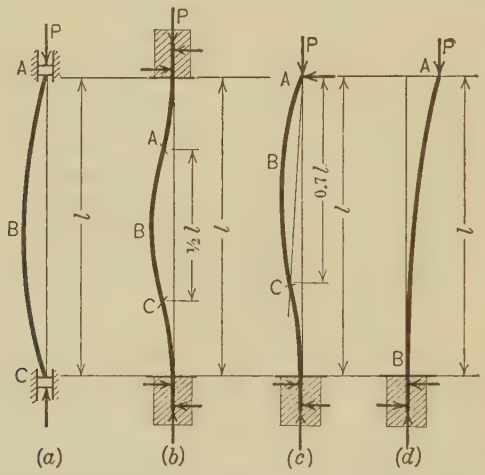

Fig. 198.—Effect of end-conditions on ideal slender columns.

same type as that in Fig. 198(*a*). Thus the maximum load P for a fixed-ended slender column is

$$P = \frac{\pi^2 EI}{\left(\frac{l}{2}\right)^2} = \frac{4\pi^2 EI}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

Therefore, a fixed-ended slender column of length l will carry as great a load as a pivot-ended column of length $\dfrac{l}{2}$; that is, a fixed-ended slender column of a given length is four times as strong as it would be with pivot ends.

In Fig. 198(*c*) the inflection point C is at a distance of approximately

$0.7l$ from A, and the part of the column ABC is of the same type as that of Fig. 198(a). Hence,

$$P = \frac{\pi^2 EI}{(0.7l)^2} = \frac{2\pi^2 EI}{l^2} \text{ approximately. . . . . (132)}$$

Therefore a slender column with one end fixed and the other end pivoted is approximately twice as strong as it would be if it had both ends pivoted.

In Fig. 198(d) the curve AB assumed by the column corresponds to the portion AB of Fig. 198(a) and hence Euler's equation for the column in Fig. 198(d) is

$$P = \frac{\pi^2 EI}{(2l)^2} = \frac{1}{4}\frac{\pi^2 EI}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

Therefore, a slender column fixed at one end and free or unrestrained at the other is only one-fourth as strong as a pivot-ended column of the same length.

II. On Columns as Used in Structures and Machines.—The end connections of columns used in structures and machines do not as a rule permit the ideal conditions discussed above to occur. The more common types of structural columns are:

Pin-ended Columns.—A column in a pin-connected structure is usually assumed to approximate closely to a pivot-ended column as regards bending in a plane perpendicular to the axis of the pins, and to approximate more or less closely to a fixed-ended column as regards bending in a plane parallel to the axis of the pins (see discussion under fixed-ended columns below). Tests [4] indicate that the friction of the pin, particularly when the pin is relatively large and the load relatively small, causes the column to act as a fixed-ended column, but when the column is subjected to larger loads it acts as a pivot-ended column, and hence in practically all cases the friction should be neglected. Pin-ended columns are commonly used for compression members in bridges, for connecting rods of engines, etc.

Columns with Ends Partially Restrained.—Columns in buildings, bridges, cranes, etc., are frequently fastened to other members by riveted or welded connections but columns with these kinds of end connections cannot have fixed ends since the connections and also the supports to which the column is attached are never rigid. The amount

[4] See tests made by James Christie, Trans. A.S.C.E., 1883, pages 85–122; also "Tests of Large Bridge Columns" by Griffith and Bragg, Technologic Paper No. 101, Bureau of Standards, 1918.

of restraint depends on the relative stiffness of the column end-connections and of the supports to which the column is attached, the stiffer the connections and supports the more the column is restrained at its ends.

As a rule it is impossible to determine in any specific case to what extent the ends are restrained, and in engineering practice the assumption is frequently made that all columns have pivot-ends. However, if the column is known to be well restrained allowance may be made for the increase in strength.

Flat-ended or Square-ended Columns.—This type of column bears against approximately plane surfaces of end supports or footings. If the surfaces of the ends of the column and of the footings were true plane surfaces and made perfect contact over the whole end surface of the column, flat-ended columns would approach closely to fixed-ended columns provided that the end supports were relatively rigid. However, there is always much uncertainty as to the way the column bears against the end supports, the unevenness of bearing causing, at working loads, an eccentricity of loading, and hence a so-called flat-ended column is, as a rule, considered to be little if any stronger than a pivot-ended column of the same slenderness ratio. Timber columns are commonly used with so-called flat ends.

PROBLEMS

307. Solve Prob. 299 assuming that the column is an ideal fixed-ended column.

308. Solve Prob. 300 assuming that the column is an ideal fixed-ended column.

309. Solve Prob. 303 assuming that the column is an ideal column having one end fixed and the other end free from restraint.

74. Methods of Obtaining Formulas for Columns Having Intermediate Slenderness Ratios.—The results of many tests of columns show that the formulas represented in Fig. 196 are substantially correct for small and large values, respectively, of $\frac{l}{r}$ but for intermediate values of $\frac{l}{r}$ (corresponding to the part of the curve between E and F say) the test results for $\frac{P}{a}$ fall below the curve, the cause of the lower values of $\frac{P}{a}$ being due to the conditions stated in Art. 68. Now the columns commonly used in engineering structures and machines have these intermediate values of $\frac{l}{r}$, and the conditions stated in Art. 68 have, in general,

a relatively greater influence on the strength of columns with intermediate slenderness ratios than on columns with either small or large values of $\frac{l}{r}$. Further, as noted in Art. 67, it is practically impossible to obtain a rational column formula that gives quantitative expression to the influences of these conditions, and hence formulas for columns of intermediate slenderness ratios are empirical. The main methods that have been used for developing such formulas are as follows:

1. To assume that the conditions stated under (2), (3) and (4) of Art. 68 will cause the column to bend as a whole when subjected to an axial load P, the maximum bending moment, at the center of the column, being $P\Delta$, where Δ is the deflection of the column. The maximum stress in the column is considered to be the sum of the direct compressive stress and the bending stress due to the moment $P\Delta$. The value of the deflection, being unknown, is eliminated from the equation by assuming that the relation between the stress and deflection in the column follows the same law as does stress and deflection in a beam. The widely used Gordon-Rankine formula, discussed in Art. 75, is obtained by this method; it must be regarded as essentially an empirical formula, for, although the statement that the stress in the column is due to combined direct compression and bending is essentially correct, the assumption made as to the relation between stress and deflection is untenable. In fact the bending moment at any section is not due mainly to the deflection of the member but to crookedness of the member and to the eccentricity of the loads, and further, the quantitative measure of the influence of the conditions stated in Art. 68 on the stress in the column is determined from experimental results.

2. To determine from test data an equation which represents the average of test results without any attempt at a rational analysis of column action. This method gives purely an empirical formula; the widely used straight-line formula discussed in Art. 77 and the parabolic formula are obtained in this way.

3. To assume that the combined effects of the conditions stated in Art. 68 are equivalent, in producing stress, to an eccentricity, e, of loading, called an *equivalent* eccentricity. The bending moment, then, after the column bends is $P(e + \Delta)$ instead of $P\Delta$ as used in Euler's equation. An expression for P is then found, from the elastic curve equation, which contains the maximum unit-stress in the column and the equivalent eccentricity e, a value for e being chosen so that the formula is made to agree with experimental results. This method leads to a modified Euler formula called the secant formula which is discussed in Art. 82. Further investigations have shown that the application of Euler's

formula $\frac{P}{a} = \frac{\pi^2 E}{(l/r)^2}$ can be extended also to shorter columns in which $\frac{P}{a}$ is above the proportional limit of the material, by substituting, in place of the constant modulus E, a reduced modulus E_1 that varies with the slenderness ratio of the column. This reduced modulus can be determined from the compression stress-strain diagram of the material.[5]

75. Gordon-Rankine Formula.—Gordon and Rankine in obtaining the formula that bears their names assumed that the column deflects like a beam as indicated in Fig. 199 when subjected to an axial load, and that the bending moment at any section is the result of this deflection. It is now known, however, that in columns of intermediate slenderness ratios the bending moment at any section is due mainly to eccentricity of loading and crookedness of the column rather than to column deflection. The resulting formula, therefore, can hardly be considered to be rational as is sometimes claimed. Nevertheless it will be desirable to follow the reasoning employed in the original derivation because it emphasizes the fact that the stress in a column is due to a direct stress and a bending stress which is a correct and important idea, although the bending moment causing the bending stress can not be calculated.

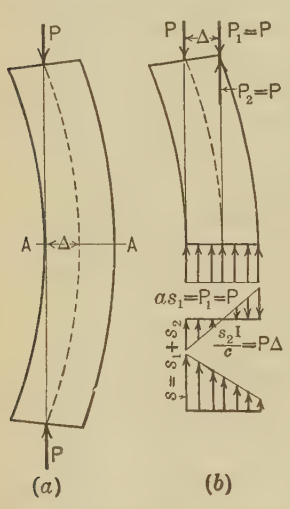

FIG. 199.—Stress in column.

The maximum value of the unit-stress is found as follows: The maximum stress occurs on section AA (Fig. 199a) where the moment is maximum and equal to $P\Delta$ and the stresses on this section must hold the external force P in equilibrium. Now, by introducing two equal and opposite forces, P_1 and P_2, each equal to P, the force P may be resolved into a force P_1 (Fig. 199b) acting through the centroid of the central section AA, and a bending couple $P\Delta$. If the force P_1 were acting alone the unit-stress s_1 developed on the area a of the section AA would be uniformly distributed over the area, and hence

$$as_1 = P, \quad \text{or} \quad s_1 = \frac{P}{a}.$$

And if the bending couple $P\Delta$ were acting alone, a resisting moment

[5] See Timoshenko's Strength of Materials, Vol. II, Art. 37–38.

$\frac{s_2 I}{c}$ would be developed, the unit-stress varying as shown in Fig. 199(*b*); this resisting moment would hold the external moment $P\Delta$ in equilibrium, and hence

$$P\Delta = \frac{s_2 I}{c}, \quad \text{or} \quad s_2 = \frac{P\Delta c}{I}.$$

The total unit-stress, then, at any point of the area, would be the algebraic sum of the unit-stresses due to P_1 and $P\Delta$. Hence the maximum unit-stress s is

$$s = s_1 + s_2 = \frac{P}{a} + \frac{P\Delta c}{I}, \quad . \quad . \quad . \quad . \quad . \quad (134)$$

and since $I = ar^2$ where r is the radius of gyration, the above equation may be written

$$s = \frac{P}{a}\left(1 + \frac{\Delta c}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

Now Δ is eliminated from eq. (135) by assuming (erroneously) that it bears the same relation to the stress in the column as exists between the deflection and stress in beams.[6] This relation for beams (see Art. 49) is $\Delta = k\frac{sl^2}{c}$ in which k is a constant depending on the kind of material of which the beams are made, the type of loading, and the end-conditions. And this may be written $\Delta = \phi\frac{l^2}{c}$ where ϕ is a constant similar to k, provided that the beams are subjected to the same maximum unit-stress and that this stress does not exceed the proportional limit of the material. Therefore Δ in eq. (135) is replaced by $\phi\frac{l^2}{c}$; the resulting equation is

$$s = \frac{P}{a}\left(1 + \phi\frac{l^2}{r^2}\right) \quad \text{or} \quad \frac{P}{a} = \frac{s}{1 + \phi\left(\frac{l}{r}\right)^2}. \quad . \quad . \quad (136)$$

This equation is called the Gordon-Rankine column formula; Gordon

[6] This assumption would do much to destroy the rational basis of the formula even if the moment in the column was mainly the result of the deflection of the column, since in a beam the stress is due to bending alone, and hence even if the column were an ideal one, only that part of the stress $\left(s - \frac{P}{a}\right)$ which is due to the bending moment $P\Delta$ should be considered to be proportional to Δ instead of the total unit-stress s.

used the least lateral dimension of the cross-section and Rankine, about 1860, introduced the radius of gyration.

A value of ϕ for columns of a given material and type of end connection is found from results of tests of such columns, the value of ϕ being selected so that the value of P in the equation agrees, as well as may be, with the average of the experimental values of the loads that caused the columns to fail, and s is the ultimate strength of a short compression block since when $\frac{l}{r}$ is small $\frac{P}{a} = s$.

The formula

$$\frac{P}{a} = \frac{50{,}000}{1 + \frac{1}{18{,}000}\left(\frac{l}{r}\right)^2}, \quad \ldots\ldots \quad (137)$$

has been widely used for structural steel columns having intermediate slenderness ratios, in which P is the axial load causing the column to fail.

It is sometimes stated that Rankine's formula is applicable to columns of all lengths. This idea may, however, lead to serious errors. For example, if eq. (137) is reliable for columns having intermediate slenderness ratios and is applied to a slender steel column having an $\frac{l}{r}$ of 240, the value of $\frac{P}{a}$ is found to be 11,910 lb./in.^{2}, whereas, the value of $\frac{P}{a}$ from Euler's formula is 5140 lb./in.^{2} And since the column undoubtedly would fail by buckling, Rankine's formula is decidedly in error on the side of danger. The limiting value of $\frac{l}{r}$ in any specified Rankine's formula should therefore be given. No single set of values of s and ϕ will render the formula accurate for all values of $\frac{l}{r}$.

Working Formulas.—A working value for s ranging from 12,500 to 16,250 lb. per sq. in. is frequently specified for structural steel columns depending somewhat on the value used for ϕ, a higher value of s calling for a higher value of ϕ. For example the following formulas are employed for structural steel columns used in buildings and similar structures.

$$\underset{(a)}{\frac{P}{a} = \frac{12{,}500}{1 + \frac{1}{36{,}000}\left(\frac{l}{r}\right)^2}}, \quad \underset{(b)}{\frac{P}{a} = \frac{14{,}500}{1 + \frac{1}{13{,}500}\left(\frac{l}{r}\right)^2}}, \quad \underset{(c)}{\frac{P}{a} = \frac{16{,}250}{1 + \frac{1}{11{,}000}\left(\frac{l}{r}\right)^2}}. \quad (138)$$

If the maximum usable compressive strength of structural steel is considered to be the compressive yield point, which is about 40,000 lb./in.^{2}, then the factors of safety for the values of s in the above formulas range from 3.2 to 2.4.

Effect of Restrained Ends.—The results of tests of built-up columns having intermediate values of slenderness ratios indicate that the type of end conditions may have a much less effect on the strength of the columns than do the deviations from the assumed ideal conditions (see Art. 68), such as eccentricity of loading, crookedness in the column as a whole, local kinks, initial stresses, variations in the properties of the material, etc. For example, some tests of steel columns having slenderness ratios less than 100 show that the strengths of the columns may be influenced but little by any of the end conditions commonly used.

Therefore, except for columns that are relatively slender and for columns that approach closely to ideal columns as may be the case for members with solid symmetrical sections, it is well to assume that the value of ϕ and s are the same for the different conditions of ends and equal to those given in the above formulas for pivot-ended columns.

Ritter's Rational Constant.—Ritter proposed that a value of ϕ be derived from the constants of the material instead of from test data. Thus, since the value of $\frac{P}{a}$ from Euler's equation should be equal, when $\frac{l}{r}$ is large, to that from Rankine's equation, we have,

$$\frac{P}{a} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{s_e}{1 + \phi\left(\frac{l}{r}\right)^2}; \quad . \quad . \quad . \quad . \quad . \quad (139)$$

the value of s_e being the proportional limit of the material since a slender column, due to the large deflection, is on the point of failing when, or even before, the proportional limit is reached (see Fig. 195). Now since unity is small compared with $\phi\left(\frac{l}{r}\right)^2$ it may be neglected and hence

$$\phi = \frac{s_e}{\pi^2 E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

which is Ritter's so-called rational constant for use in Rankine's formula for pivot-ended columns. This constant, however, in no way changes the empirical nature of Rankine's formula.

76. Significance of Assumption That Column Composed of Component Parts Acts as a Unit.—All column formulas contain the slenderness ratio $\frac{l}{r}$ in which r is the radius gyration of the *whole* cross-sectional

area of the column about the centroidal axis of the section. And, in order to make r large so that $\frac{l}{r}$ shall be small, columns composed of component parts (built-up columns) such as two channels latticed together (Fig. 201), etc. may have the component parts placed so far apart that they act independently and fail individually without permitting the full strength of the column acting as a unit to develop as is assumed in the design formulas.

To what extent the cross-section may thus be distended without causing the column to fail because the parts do not act as a unit cannot be definitely known, but it is probable that some of the failures of columns have been due to this condition. It is common practice to design built-up columns so that the slenderness ratio of each unsupported part of the built-up column shall not exceed the $\frac{l}{r}$ of the whole column.

ILLUSTRATIVE PROBLEM

Problem 310.—The steel parallel rod AB (Fig. 200) of a locomotive has a rectangular cross-section 1 in. by 3 in. and is 6 ft. long. When the locomotive is starting, the rod acts mainly as an axially loaded column, the cross-bending due to the weight and to the centrifugal forces being negligible. The rod may be considered to be a pin-ended column as regards bending in the vertical plane and a fixed-ended column as regards bending in a horizontal plane; due to the shape of cross-section, the bar tends to bend in the horizontal plane, but the restraint of the pins tends to cause it to bend in a vertical plane as a pin-ended column. Find the working load to which the rod should be subjected, using a working value for s equal to one-fourth of the yield-point; assume a value for ϕ equal to $\frac{4}{25{,}000}$ for the pin-ended column and hence $\frac{1}{25{,}000}$ for fixed ends.

Solution.—The yield-point will be assumed to be 40,000 lb. per sq. in. and hence the working value of s is 10,000 lb. per sq. in.

(*a*) Treating the rod as a pin-ended column:

$$r^2 = \frac{I_x}{a} = \frac{\frac{1}{12}bd^3}{bd} = \frac{1}{12}d^2 = \frac{1}{12}(3)^2 = 0.75 \text{ in.}^2,$$

whence,

$$\left(\frac{l}{r}\right)^2 = \frac{(72)^2}{0.75} = 6910 \quad \text{and} \quad \frac{l}{r} = 83.2,$$

$$P = \frac{as}{1 + \phi\left(\frac{l}{r}\right)^2} = \frac{3 \times 10{,}000}{1 + \frac{4}{25{,}000}(83.2)^2}$$

$$= \frac{30{,}000}{1 + 1.106} = 14{,}250 \text{ lb.}$$

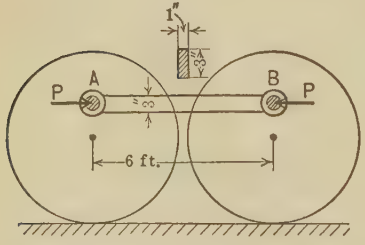

FIG. 200.—Parallel-rod in compression.

(*b*) Treating the rod as a fixed-ended column:

$$r^2 = \frac{I_y}{a} = \frac{\frac{1}{12}db^3}{db} = \frac{1}{12}b^2 = \frac{1}{12}\text{ in.}^2, \left(\frac{l}{r}\right)^2 = \frac{(72)^2}{\frac{1}{12}} = 62{,}300 \quad \text{and} \quad \frac{l}{r} = 249,$$

$$P = \frac{30{,}000}{1 + \frac{1}{25{,}000}(249)^2} = \frac{30{,}000}{1 + 2.48} = 8630 \text{ lb. working load,}$$

and hence the maximum working load the column should resist is 8630 lb. If, however, the ends can not be regarded as fixed with respect to bending in the horizontal plane, because of loose bearings, the value of ϕ should be more nearly the same for the two cases. But the safe load should then be determined also from Euler's formula for round ends since with the large value of l/r the column would probably fail by buckling.

PROBLEMS

311.—A latticed, pin-ended, axial-loaded steel column is made of two 10-in. 15.3-lb. channels (Fig. 201). The distance, h, between backs is such that the moments of inertia about the x and y axes are equal. The column is 20 ft. long. Calculate the slenderness ratio and the working axial load for the column. Use formula (138*c*).

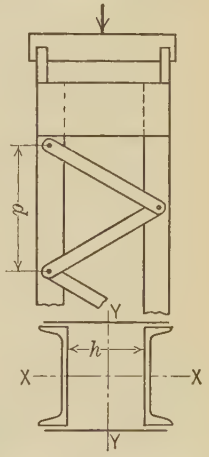

FIG. 201.—Latticed column.

312. A low-carbon steel bar 8 ft. long and 2.4 in. in diameter is to be tested as a pivot-ended column, the load being applied axially. The yield-point of the material is found to be 36,000 lb. per sq. in. (*a*) Calculate the slenderness ratio and the maximum load P, by Rankine's formula (137), the column would be expected to resist. (*b*) Also find P by use of the Euler formula (see discussion following formula (137).

Ans. (*a*) $\frac{l}{r} = 160$, $P = 93{,}300$ lb.

313. Two 8-in. 25.5-lb. steel I-beams are latticed together, the distance between them being such that the moments of inertia of the cross-section about the two axes of symmetry are equal. The column is pin-ended and is 20 ft. long. Calculate the value of $\frac{l}{r}$ of the column and the maximum value of s accompanying an axial load of 40 tons. Assume $\phi = \frac{4}{25{,}000}$.

Ans. $\frac{l}{r} = 79.3$; $s = 10{,}780$ lb. per sq. in.

314. What should be the spacing of 4 × 4 in. timber posts 6 ft. long, that support a horizontal platform which is loaded with a uniform load of 100 lb. per sq. ft.? Assume the platform to have no lateral support and hence assume the columns to be fixed at the lower end and free at the upper end; the value of ϕ for such a column will be assumed to be four times that for a pivot-ended column (see Art. 73). Assume a working value of s of 1000 lb. per sq. in., and ϕ for pin ends $= \frac{1}{800}$.

315. The steel connecting-rod on a certain engine is 3 in. by $1\frac{1}{4}$ in. in cross-section and 2 ft. long. It may be assumed to be a pin-ended column for bending about an axis perpendicular to the 3-in. side and a fixed-ended column for bending about an axis parallel to the 3-in. side. What is the greatest pressure that may be applied on the 20-in. piston without exceeding a value for s in Rankine's formula of 16,000 lb. per sq. in.? Assume the connecting rod to be in its horizontal position. Use values of ϕ given in Problem 310. *Ans.* 162 lb. per sq. in.

316. Would eq. (137) be expected to give a satisfactory answer to Prob. 299? Solve the problem by use of eq. (137) and compare the result with that found in the solution of Prob. 299.

317. Assuming the compressive yield point of structural steel to be 40,000 lb./in.^{2}, plot a graph like that in Fig. 196 and then in the same diagram plot a graph of eq. (137) and observe whether this formula gives reliable values for $\frac{P}{a}$ for small and large values of $\frac{l}{r}$.

318. The member BC of the pin-connected truss shown in Fig. 202 is made of two 6-in. 15.5-lb. channels latticed together (see Fig. 201); the moments of inertia about the two principal axes are equal. Calculate the value of s in Rankine's formula (the so-called unit-stress) in the member, using ϕ as $\frac{4}{25{,}000}$.

Ans. $s = 7800$ lb. per sq. in.

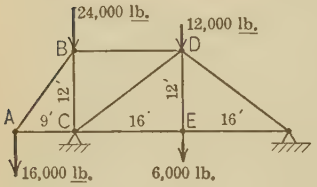

FIG. 202.—Truss containing compression members.

319. Find the so-called unit-stress in member AC of Fig. 202, assuming it to have the same cross-section as that of member BC described in the preceding problem.

320. A round timber column is 12 in. in diameter and 20 ft. long. If the column has fixed ends, what working axial load may be applied to the column assuming a working stress of 800 lb. per sq. in., and a value of ϕ of $\frac{1}{3000}$.

Ans. $P = 28{,}800$ lb.

321. A hollow cast-iron column has an outside diameter of 8 in., and inside diameter of 6 in. and a length of 15 ft. If the ends are fixed what is the so-called unit-stress in the column when it carries an axial load of 80,000 lb.? Use a value of ϕ of $\frac{1}{5000}$.

NOTE.—The problems after Art. 77 may be solved by Rankine's formula if additional problems are desired.

77. Straight-line Formula.—If a series of columns, all made of the same material, with the same general shape of cross-section and the same end conditions but with different slenderness ratios, are tested to failure

by subjecting them to the same type of loading, the test results will show that, *for columns having intermediate slenderness ratios,* the load per unit area $\left(\frac{P}{a}\right)$ decreases approximately in the same proportion that the slenderness ratio increases. In other words the relation between $\frac{P}{a}$ causing failure and $\frac{l}{r}$ is represented approximately by a straight line.

For example, Fig. 203 shows the experimental results obtained from tests [7] of pin-ended axially-loaded structural-steel columns having H-

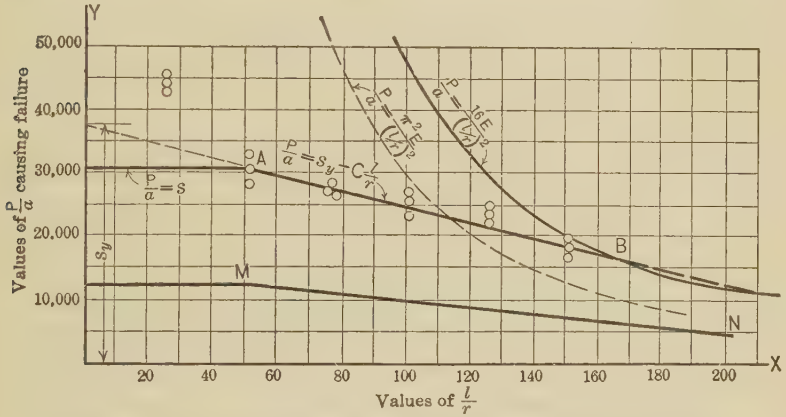

FIG. 203.—Graphical representation of straight-line column formula.

sections. The ordinates represent loads causing failure divided by the cross-sectional area $\left(\frac{P}{a}\right)$ and the abscissas represent the slenderness ratios of the columns $\left(\frac{l}{r}\right)$. The variation [8] in the results for the three columns in each group indicates the effects of the factors (accidental eccentricities, initial crookedness, initial stresses, etc.) discussed in Art. 68. As shown in Fig. 203, for values of $\frac{l}{r}$ between 50 and 150 approximately, a sloping straight line fits the test results fairly well; for values of $\frac{l}{r}$ less than about 50 the load per unit-area causing failure is con-

[7] Made at the Watertown Arsenal; see "Tests of Metals," 1909.

[8] The variations in some tests are much greater than those shown in Fig. 203, particularly in test of large built-up sections.

sidered to be constant and equal approximately to the yield-point of the material, even though laboratory tests of columns *having solid sections* sometimes show (as in Fig. 203) that the actual $\frac{P}{a}$ causing failure is greater than the yield-point of the material; tests of built-up columns, however, rarely give values of $\frac{P}{a}$ greater than the yield-point.

Further, the values of $\frac{P}{a}$ found from Euler's formula, $\frac{P}{a} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$, for ideal pivot-ended columns are less[9] than the values of $\frac{P}{a}$ found from tests, when $\frac{l}{r}$ is relatively large (170 to 200). T. H. Johnson[10] found from a careful study of the results of tests of slender columns that if π^2 in Euler's equation were replaced by 16, the resulting equation, $\frac{P}{a} = \frac{16E}{\left(\frac{l}{r}\right)^2}$, would agree well with the test results, for pin-ended slender columns. The curve representing this equation is shown in Fig. 203. Thus, the straight line AB in Fig. 203 represents the test results only between certain limiting values of $\frac{l}{r}$.

Now the equation of a straight line (similar to AB, Fig. 203) is

$$y = -Cx + s_y,$$

in which C is the slope of the line, being negative in Fig. 203, and s_y is the intercept on the y-axis. Therefore the equation of the line AB (Fig. 203) is

$$\frac{P}{a} = s_y - C\frac{l}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

which is called the straight-line column formula; it was first proposed by Mr. T. H. Johnson[10] in 1886.

[9] This is probably due to the fact that a pin-ended column does not act as an ideal pivot-ended column; the friction of the pin exerts an end-moment which causes the column to act with an end restraint intermediate between a pivot-ended and a fixed-ended column; the influence of this end restraint being relatively large in the relatively slender columns.

[10] See Transactions of American Society of Civil Engineers, Vol. 15, 1886, p. 517.

But as already noted, this equation never reduces to $\frac{P}{a} = s_y$ since the value of $\frac{l}{r}$ in the equation cannot be less than that corresponding to the point A (Fig. 203).

The above equation for the straight line shown in Fig. 203 is

$$\frac{P}{a} = 37{,}500 - 125\frac{l}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

and this equation agrees reasonably well with most of the available test data [11] obtained from tests on structural-steel pin-ended, axially-loaded columns in which the full strengths of the columns were developed; that is, the columns did not fail due to local wrinkling or to weak end connections, etc.

Since the value of $\frac{P}{a}$ given by equation (142) is the load per unit-area that causes failure, a working value of $\frac{P}{a}$ may be found by dividing the right-hand member of the equation by a factor to give a margin of safety—a so-called factor of safety. A value of 2.5 is recommended for structural steel columns by a joint committee of the American Railway Engineering Association (A.R.E.A.) and the American Society of Civil Engineers (A.S.C.E.); the working load per unit-area then, is

$$\frac{P}{a} = 15{,}000 - 50\frac{l}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

with a specified *maximum* allowable value for $\frac{P}{a}$ of 12,500 lb. per sq. in. This equation is represented by the line MN in Fig. 203. Thus, the design formulas for structural-steel pin-ended [12] axially-loaded columns so designed that they do not fail due to local wrinkling or to weak end-connections are:

$$\frac{P}{a} = 12{,}500 \quad \text{if } \frac{l}{r} \text{ is not greater than 50}, \quad . \quad . \quad . \quad . \quad (144)$$

[11] See Report of Sub-Committee on Iron and Steel Structures of the American Railway Engineering Association, Vol. 21, Jan., 1920. Appendix B.

[12] The effect of end conditions is discussed in Art. 73; but specifications usually make no allowance for the end restraint that occurs due to riveted end-connections as used in buildings, bridges, cranes, etc.

$$\frac{P}{a} = 15{,}000 - 50\frac{l}{r} \quad \text{if } \frac{l}{r} \text{ lies between 50 and 150} \quad . \quad (145)$$

$$\frac{P}{a} = \frac{6.4E}{\left(\frac{l}{r}\right)^2} \quad \text{if } \frac{l}{r} \text{ is greater than 150.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

The values of $\frac{P}{a}$ found from the last two of the above equations differ but little for values of $\frac{l}{r}$ between 150 and 210, as indicated in Fig. 203, and hence the second of the above equations may be considered fairly reliable for values of $\frac{l}{r}$ between 50 and 210. In specifications, however, the value of $\frac{l}{r}$ is frequently limited to 120 for main members and to 200 for secondary members.

The following formulas were formerly given in the specifications of the A.R.E.A. and are still widely used for the design of columns in buildings, etc.

$$\frac{P}{a} = 14{,}000 \text{ if } \frac{l}{r} \text{ is less than 50,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144a)$$

$$\frac{P}{a} = 16{,}000 - 70\frac{l}{r} \text{ if } \frac{l}{r} \text{ is between 50 and 120} \quad . \quad . \quad (145a)$$

Thus the later specifications (eq. 144 and 145) change the former (eq. 144*a* and 145*a*) by reducing the maximum permissible unit-stress and by deducting less for the effect of length. It is important to note that the reduction in the permissible stress in columns has been accompanied by a tendency to increase the permissible stresses in beams and tension members, and this is consistent with the discussion in Arts. 67 and 68.

In designing a column, by the straight-line formula, to resist a given load, the dimensions of the cross-section may easily be found directly from the formula, provided that the area is circular or square, etc., for which a and r may be expressed in terms of a single variable, but the dimensions of sections of columns made of rolled shapes are found by the method of trial and error. Steel companies' hand books give safe loads on columns having various sections as calculated by different formulas, and these values may be used to secure the first trial value of the section when employing a formula different from that used in the handbook.

Timber and Cast-iron Columns.—The following straight-line formulas for axially loaded columns giving working values of $\frac{P}{a}$ are representative

of those in common use; see engineers' hand books for other formulas.

$$\frac{P}{a} = 1{,}000 - 6\frac{l}{r} \quad \text{for structural timber,} \quad (147)$$

$$\frac{P}{a} = 9{,}000 - 40\frac{l}{r} \quad \text{for cast iron.} \quad (148)$$

$$\frac{P}{a} = 15{,}000 - 123\frac{l}{r} \quad \text{for one of the stronger structural aluminum alloys and for } \frac{l}{r} \text{ up to 81.} \quad (149)$$

ILLUSTRATIVE PROBLEM

Problem 322.—Select a steel I-section 9 ft. long to carry an axial compressive load of 80,000 lb., assuming the column to have round ends.

Solution.—An approximate value of the area of section needed may be found by considering the member to be a compression block, in which case $\frac{P}{a} = 12{,}500$, and hence $a = \frac{80{,}000}{12{,}500} = 6.4$ in.^{2} Now by consulting a steel maker's hand book for properties of rolled steel sections we find that the minimum radius of gyration for I-sections having areas of about 7 in.^{2} is not far from 1.0, and hence $\frac{l}{r}$ is approximately 108, which means that the a in $\frac{P}{a} = 15{,}000 - 50\frac{l}{r}$ must be considerably larger than 6.4 in.^{2}

As a second trial, select a 10-in. 25.4-lb. I-section for which $a = 7.38$ in.^{2} and $r_{\text{min.}} = 0.97$ in. Then $\frac{l}{r} = \frac{108}{0.97} = 111$. Hence

$$\frac{80{,}000}{a} = 15{,}000 - 50 \times 111, \quad \text{or} \quad a = 8.47 \text{ in.}^2$$

Therefore a section having an area larger than 7.38 in.^{2} is needed. As a further trial select a 10-in. 30-lb. I-section, for which $a = 8.75$ in.^{2}, $r_{\text{min.}} = 0.93$ in., and hence $l/r = 116$. Then

$$a = \frac{80{,}000}{15{,}000 - 50 \times 116} = 8.70 \text{ in.}^2$$

Therefore a 10-in. 30-lb. I-section having an area of 8.75 in.^{2} is satisfactory.

PROBLEMS

323. A standard 12-in. 40.8-lb. I-beam 9 ft. long is used as a column with riveted end-connections that are assumed to be equivalent to pin-ends. What working axial load should it carry? *Ans.* 118,400 lb.

324. A square timber column 16 ft. long resists a working axial load of 120,000 lb. Find the dimensions of the cross-section. *Ans.* 13.1 in. square.

325. A steel column in a certain power-house has an unsupported length of 22 ft. and is made up of a 12-in. by $\frac{1}{2}$-in. plate, to which are riveted four 5-in. by $3\frac{1}{2}$-in. by $\frac{1}{2}$-in. angles as shown in Fig. 204. Find the working axial load.

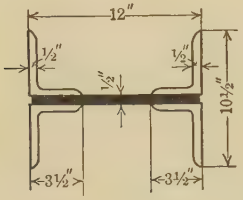

FIG. 204.

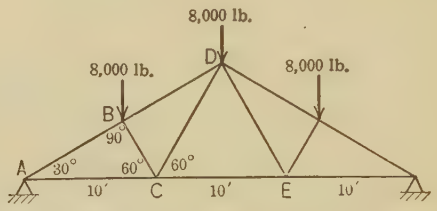

FIG. 205.—Truss containing compression members.

326. What size of equal leg angle is required for strength for the member *BC* of the truss shown in Fig. 205 (see also Problem 68 and Fig. 49). Use the Joint Committee formula (143).

327. Solve Prob. 311 by use of the Joint Committee straight-line formula (143).

328. The steel column whose cross-section is shown in Fig. 210 is 12 ft. long and acts as a pin-ended column with the axis of the pin parallel to the 2–2 axis. The column is subjected to an axial load of 286,000 lb. If you were asked to report on the safety of the column, what would you report? Show calculations.

FIG. 206.

329. The upper chord member of a truss (similar to *AB* or *BD* in Fig. 205) consists of two angles, separated $\frac{3}{8}$ in. by washers (Fig. 206). If the unsupported length of the chord is 5 ft. and the total compressive axial stress is 65,000 lb. find the size of angles required. Assume that the two angles act as a unit.

Ans. 5-in. by $3\frac{1}{2}$-in. by $\frac{5}{16}$-in. angles.

330. An aluminum 12-in. 14.5-lb. I-section is used as a column 6 ft. long with round ends. What working axial load should it carry? The principal moments of inertia are $I_1 = 229$ in.^{4} and $I_2 = 9.87$ in.^{4}; and the area of cross-section is 10.3 in.^{2}

Ans. 61,300 lb.

NOTE.—The problems after Art. 75 may be solved by the straight-line formula if additional problems are desired.

78. Straight-line Formulas for Fixed-ended Columns.—As noted in Art. 73 the strength of an axially loaded slender ideal column with fixed or flat ends is four times that of a similar column having pivot ends. The strength of an ideal column having a very small slenderness ratio, on the other hand, is approximately the same for fixed ends as for pivoted ends since the strength of the column depends only on the strength of the material. Further, since the fixing of the ends increases the resistance of the column to bending chiefly, the influences of fixed ends

would be expected to increase with the increase of $\frac{l}{r}$, and test results [13] indicate that this is true.

If test results for $\frac{P}{a}$ and $\frac{l}{r}$ for *fixed-ended* structural-steel columns are plotted, in a way similar to that used in Fig. 203 for pin-ended columns, a straight line is found to fit approximately the average of the test results, the equation of the line being

$$\frac{P}{a} = 37{,}500 - 100\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

If a reduction factor of 2.5 is used, as was done with the equation for pin-ended columns, the resulting equation giving the working load for structural-steel fixed-ended columns is

$$\frac{P}{a} = 15{,}000 - 40\frac{l}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

in which $\frac{P}{a}$ must not exceed 12,500 lb. per sq. in.; thus for values of $\frac{l}{r}$ less than 62.5 the equation is, $\frac{P}{a}$ = 12,500 lb. per sq. in. The above equations are recommended by the American Railway Engineering Association. The upper limit for $\frac{l}{r}$ to be used in eq. (151) is about 200 and for larger values than 200 Euler's equation,[14] with a suitable reduction factor, is applicable.

Equation (151) should be used only in cases where the restraint of the end connections are comparable to that offered by the relatively rigid heads of a testing machine when making a good contact with the end surfaces of the column.

The Structural Aluminum Handbook gives

$$\frac{P}{a} = 15{,}000 - 61.5\frac{l}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

[13] See Bulletin American Railway Engr. Assoc., Vol. 21, Jan., 1920, Appendix B.

[14] Mr. T. H. Johnson found from a careful study (Trans. Am. Soc. C. E., Vol. 15, 1886, p. 517) of test data for slender columns having flat ends that approached closely to fixed ends, that Euler's equation was approximately $\frac{P}{a} = \frac{25E}{\left(\frac{l}{r}\right)^2}$ instead of $\frac{P}{a} = \frac{4\pi^2 E}{\left(\frac{l}{r}\right)^2}$ as found for *ideal* slender fixed-ended columns.

for fixed-ended columns made of one of the stronger aluminum alloys, where $\frac{l}{r}$ must not exceed 162. This is to be compared with

$$\frac{P}{a} = 15{,}000 - 123\frac{l}{r},$$

for pivot-ended columns of the same material, where $\frac{l}{r}$ must not exceed 81.

79. Straight-line Formula for High Carbon and Alloy Steel Columns.—Formulas 144, 145 and 151 apply to columns made of steel having an ultimate tensile strength of 55,000 to 65,000 lb. per sq. in. and a yield-point of at least one-half of the ultimate strength, since these are the values stated in the specifications of steel for most structural purposes.

Now if steel having a higher strength than this is used, the strengths of columns having small values of $\frac{l}{r}$ will be increased in about the same ratio as the yield-points (or proportional limits) are increased, whereas the strength of the very slender columns will be increased very little, if any, since the strength of such columns depends on the stiffness (modulus of elasticity) of the material (not the strength of the material) and the stiffness of all grades of steel is practically constant.

There are at present not sufficient test data from which to determine the constants in a straight-line formula for columns of high-strength steel, but the few test results available indicate that high-strength steel is of little advantage for slender columns and that, in accordance with the above reasoning, the formula for structural steel columns should be modified for use with high-strength steel by increasing the constant C more than the constant s_y, otherwise the value of $\frac{P}{a}$ will be too high for the larger values of $\frac{l}{r}$. In other words, if the line AB (Fig. 207) represents the straight-line formula for ordinary structural steel the line CD would represent the formula for high-strength steel.

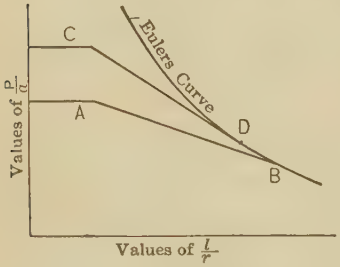

FIG. 207.

For example the safe load for a column made of high-strength steel

(having a proportional limit of about 60,000 lb. per sq. in.) is given by the following straight line equation

$$\frac{P}{a} = 22{,}500 - 90\frac{l}{r}, \quad \ldots\ldots \quad (153)$$

with a maximum value for $\frac{P}{a}$ of 18,750 lb. per sq. in., as compared to the value of $\frac{P}{a}$ in the equation $\frac{P}{a} = 15{,}000 - 50\frac{l}{r}$ (with a maximum of 12,500 lb. per sq. in.) for a column made of structural steel, as given in Art. 77.

80. Eccentrically Loaded Columns.—In the preceding articles the column was assumed to be axially loaded. Many columns, however, are subjected to eccentric loads, and the Rankine formula and the straight-line formula may be modified to take account, approximately, of the eccentricity of loading as follows (the modified Euler's equation is discussed in Art. 82):

Modified Rankine's Formula.—In Fig. 208(*a*) let the column be subjected to a load P having an eccentricity e. This load may be resolved into an axial load P_1 (equal to P) and a couple Pe (Fig. 208). Now the load P_1 if acting alone would cause a unit-stress s_1 on the concave side which, according to Rankine's formula, is considered to be

$$s_1 = \frac{P}{a}\left(1 + \phi\frac{l^2}{r^2}\right),$$

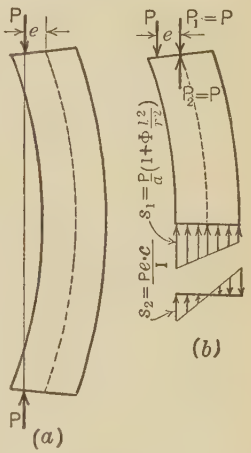

FIG. 208.—Eccentric load on column.

and the bending couple Pe if acting alone would cause, and be held in equilibrium by, a resisting moment $\frac{s_2 I}{c}\left(Pe = \frac{s_2 I}{c}\right.$, see Fig. 208*b*$\left.\right)$. The unit-stress, s, on the concave side, then, is considered to be

$$\begin{aligned} s &= s_1 + s_2 \\ &= \frac{P}{a}\left(1 + \phi\frac{l^2}{r^2}\right) + \frac{Pec}{I} \\ &= \frac{P}{a}\left(1 + \phi\frac{l^2}{r^2} + \frac{ec}{r^2}\right), \quad \ldots\ldots \quad (154) \end{aligned}$$

in which r is the radius of gyration of the cross-section about the cen-

troidal axis, and c is the distance from the centroidal axis to the fiber on which the unit-stress s occurs.

Modified Straight-line Formula.—According to the straight-line formula for axial loads (Art. 77), the maximum average unit-stress $\left(\frac{P}{a}\right)$ to which the column may be subjected is

$$\frac{P}{a} = s_y - C\frac{l}{r}. \quad \ldots\ldots\ldots \quad (155)$$

That is, a column with a given $\frac{l}{r}$ may be considered to be an axially loaded compression block in which the allowable unit-stress is $\left(s_y - C\frac{l}{r}\right)$, provided that the value of $\frac{l}{r}$ is between certain limiting values, as explained in Art. 77.

Now if the load P is applied with an eccentricity e, as in Fig. 208(a), the unit-stress due to the bending moment Pe would be

$$s_2 = \frac{Mc}{I} = \frac{Pec}{I} = \frac{Pec}{ar^2},$$

and the maximum unit-stress in the column would be increased by this amount as indicated in Fig. 208(b), and hence the value of $\frac{P}{a}$ must be reduced by the amount s_2.

Thus,

$$\left.\begin{aligned} \frac{P}{a} &= \left(s_y - C\frac{l}{r}\right) - \frac{Pec}{ar^2} \\ \text{or} \qquad \frac{P}{a}\left(1 + \frac{ec}{r^2}\right) &= s_y - C\frac{l}{r}. \end{aligned}\right\} \quad \ldots\ldots \quad (156)$$

That is, a column with a given $\frac{l}{r}$ and subjected to a load P having an eccentricity e may be treated as an axially loaded short compression block for which the allowable unit-stress is $\dfrac{s_y - C\frac{l}{r}}{1 + \frac{ec}{r^2}}$, provided that the value of $\frac{l}{r}$ is between certain limiting values as explained in Art. 77.

PROBLEMS

331 A timber column supports one end of a beam as indicated in Fig. 209. The column has a square cross-section 12 in. by 12 in. and is 12 ft. long, and is assumed to have ends equivalent to pin ends. If the load is assumed to act with an eccentricity of 4 in., what working load should the column support? Use formulas (147) and (156). *Ans.* 36,000 lb.

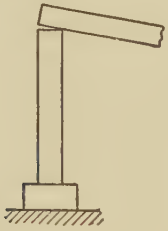

FIG. 209.—Eccentric load on column.

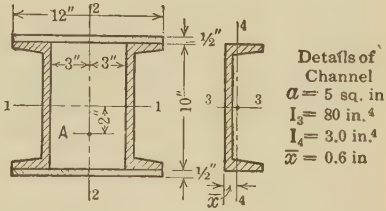

FIG. 210.

332. A pin-ended steel column 20 ft. long having the cross-section shown in Fig. 210 is subjected to a load of 40 tons having an eccentricity of 2 in.; the load acts at A, Fig. 210. Calculate the maximum value of s in formula (154) using for ϕ the value given in (138a).

333. A column is built up of two 10-in. 15-lb. channels laced together (see Fig. 201) so that the distance between backs is 6.33 in. (the moments of inertia about the two principal axes are then equal). The column is 18 ft. long. Calculate the working load if the eccentricity is (*a*) 0 in., (*b*) 2 in., (*c*) 4 in., measured along the YY axis in Fig. 201. Use Rankine's formula (138a).

81. The Parabolic Formula.—It was pointed out in 1892 by Professor J. B. Johnson that the relation between the load per unit area, $\frac{P}{a}$, causing failure as obtained from tests, and $\frac{l}{r}$ could be represented with close agreement by the equation of a parabola having its vertex at a point on the axis of loads whose ordinate is the compressive strength of the material, and being tangent to Euler's Curve. The equation is

$$\frac{P}{a} = s_y - k\left(\frac{l}{r}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

in which s_y is the compressive strength of the material (the yield point for ductile material) and $k = \frac{{s_y}^2}{4\pi^2 E}$ if the curve is tangent to Euler's curve. The student may sketch in such a parabola in Fig. 203 observing that it becomes tangent to Euler's curve at about the point B.

The Column Committee of the American Railway Engineering Association (A.R.E.A.) from a study of test data recommended the equation

$$\frac{P}{a} = 32{,}500 - \frac{5}{8}\left(\frac{l}{r}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

for structural steel pin-ended columns subjected to approximately axial loads, where $\frac{P}{a}$ is the load per unit-area at which the column fails. By applying a factor of safety of 2.5 as was done with the straight-line formula (see Art. 77) the working or design formula becomes

$$\frac{P}{a} = 13{,}000 - \frac{1}{4}\left(\frac{l}{r}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

which applies for all values of $\frac{l}{r}$ up to the value corresponding to the point of tangency to Euler's curve.

One advantage sometimes claimed for the Johnson parabola over the straight-line formula is that the area of cross-section required for a column of a certain length to carry a specified load can be obtained directly without employing a trial and error method that is practically necessary in the straight-line formula. For example, if the column is short the area required is $a_0 = \frac{P}{s_y}$. It is evident from the parabolic formula that a depends on r^2. Let $a = qr^2$; the value of q (called the shape- or form-factor) being determined from a cross-section similar to the one that it is proposed to use; tables of values of q may be prepared if this formula is to be used frequently.

Formula (157) may now be written

$$\frac{a_0 s_y}{a} = s_y - k\frac{l^2}{a}q. \quad \text{But } k = \frac{{s_y}^2}{4\pi^2 E}.$$

Therefore

$$a = a_0 + \frac{s_y}{4\pi^2 E} l^2 q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

All the quantities in this equation are known except a. The value of $\frac{s_y}{4\pi^2 E}$ for structural steel is approximately $\frac{40{,}000}{4 \times 10 \times 30 \times 10^6} = \frac{1}{30{,}000}$. This method when applied to the straight-line formula leads to a quadratic equation for a.

82. Equivalent Eccentricity. Secant Formula.—As stated in Art. 74, one method of taking account of the effect of crookedness, lack of homogeneity of material, initial stresses, unintentional or accidental eccentricity, etc., is to assume that these conditions are equivalent, in producing stress, to a positive eccentricity of loading on a straight column of homogeneous material free from initial stresses. Thus, if the eccentricity be denoted by e, the bending moment at any section of the column is $P(e + y)$, as shown in Fig. 211(a), instead of Py as was the case in the derivation of Euler's column formula for an axially-loaded column (Art. 71). Now, as shown below, this change in the expression for the bending moment leads to a modification of Euler's formula, called the *secant formula*, from which the maximum unit-stress (and also maximum deflection) caused by a given load and eccentricity can be found; the stress and deflection do not occur in Euler's equation for an axially-loaded column. Further, the secant formula, can be used for eccentrically loaded columns also, since e then is the actual eccentricity plus the equivalent eccentricity.

Secant Formula.—The secant formula may be derived as follows: Let Fig. 211(a) represent a column of length l subjected to a load P having an eccentricity e. The equation of the curve assumed by the column is the same as that of a round-ended column of length L (Fig. 211b) subjected to an axial load P, the part MN of the column (Fig. 211b) being in the same state whether the load P is applied at A with eccentricity e or at C with no eccentricity. Now, as stated in Art. 71, when the load P is applied at C as an axial load on a pivot-ended column, the column deflects when the load reaches a value P_1 (say) slightly greater than that of P given by the Euler formula $P = \dfrac{\pi^2 EI}{l^2}$. Further, it was stated in Art. 71 that although there is a definite deflection corresponding to each value of P_1, a very small increase in the load causes a very large increase in the deflection, and hence we may assume that when the column bends, its position may be that of any one of the several curves CD in Fig. 211(b). But, for any given eccentricity e and length l, as indicated in Fig. 211(b), there is only one curve that the column can assume, and the maximum deflection Δ and the maximum unit-stress due to the bending moment

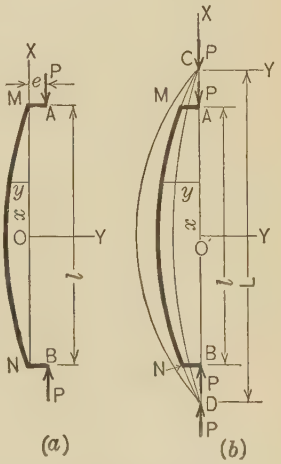

FIG. 211.—Equivalent eccentricity.

$P(e + \Delta)$ do not disappear from the elastic curve equation as was the case in the treatment of the axially-loaded slender column in Art. 71.

Now, from Art. 71, the equation of any one of the curves CD (Fig. 211b) is

$$y = \Delta \sin \sqrt{\frac{P}{EI}}\, x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (161)$$

if the axes are chosen with the origin at C; but if the axes are chosen with the origin at the center of the column length (at O', Fig. 211), the equation is

$$y = \Delta \cos \sqrt{\frac{P}{EI}}\, x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (162)$$

But y in this equation is the deflection of the pivot-ended column of length L (Fig. 211b) and hence if y is made to represent the deflection of the eccentrically-loaded column as shown, Fig. 211(a), the equation of the elastic curve assumed by the column is

$$y + e = (\Delta + e) \cos \sqrt{\frac{P}{EI}}\, x. \quad . \quad . \quad . \quad . \quad (163)$$

Now $y = 0$ when $x = \frac{l}{2}$. Therefore,

$$e = (\Delta + e) \cos \frac{l}{2} \sqrt{\frac{P}{EI}}, \quad . \quad . \quad . \quad . \quad (164)$$

and hence the maximum deflection of the eccentrically-loaded column is

$$\Delta = e\left(\sec \frac{l}{2} \sqrt{\frac{P}{EI}} - 1\right), \text{ or } \Delta = e\left(\sec \frac{l}{2r} \sqrt{\frac{P}{aE}} - 1\right). \quad . \quad (165)$$

The maximum bending moment is

$$M = P(\Delta + e) = P\left[e\left(\sec \frac{l}{2r} \sqrt{\frac{P}{aE}} - 1\right) + e\right]$$

$$= Pe \sec \frac{l}{2r} \sqrt{\frac{P}{aE}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

The maximum fiber unit-stress s is

$$s = \frac{P}{a} + \frac{Mc}{I} = \frac{P}{a} + \frac{Pec \sec \frac{l}{2r} \sqrt{\frac{P}{aE}}}{ar^2}$$

$$= \frac{P}{a}\left(1 + \frac{ec}{r^2} \sec \frac{l}{2r} \sqrt{\frac{P}{aE}}\right). \quad . \quad . \quad . \quad . \quad . \quad (167)$$

The expression $\frac{l}{2r}\sqrt{\frac{P}{aE}}$ is an angle expressed in radians and is called the *Eulerian angle.* The above equation may also be written in the form

$$P = \frac{as}{1 + \frac{ec}{r^2}\sec\frac{l}{2r}\sqrt{\frac{P}{aE}}}. \quad . \quad . \quad . \quad . \quad . \quad (168)$$

The above equations are valid only for values of s within the proportional limit of the material since the elastic curve equation and the flexure formula are used in their derivations. But within the proportional limit they are valid provided that the column acts as a unit and hence does not fail due to local wrinkling or weak end connections. And when these conditions are satisfied the equivalent eccentricity e may be selected so that the calculated values of $\frac{P}{a}$ for axially loaded columns are made to agree as well as may be with experimental results. The secant formula has been presented from time to time in technical literature as being as satisfactory a solution of column action as is likely to be obtained, but due largely to its relatively complicated form and to insufficient experimental data for determining reliable values of the equivalent eccentricity it is not in general use.

When the dimensions (a, c, r, and l) of the column are given and a value of e is known or assumed, the load P required to cause a specified unit-stress, s, may be found from equation (167). And likewise the area a needed to keep the unit-stress from exceeding the prescribed value, s, when the column is subjected to a given load, P, may be found if values of c, r and s are known. But in equation (167) since the quantities P and a are not expressed explicitly in terms of the other quantities, the solution of the equation is most conveniently made by trial and error. To aid in the solution of equation (167) values of $\frac{l}{2r}\sqrt{\frac{P}{aE}}$ for various values of $\frac{P}{a}$ and $\frac{l}{r}$ are given in Table 3 for use in the investigation of *steel* columns (E = 30,000,000 lb. per sq. in. for steel).

Values of e and of $\frac{ec}{r^2}$.—The value of the equivalent eccentricity e to be used in equation (167) for axially loaded columns must be assumed, the value depending on the judgment of the engineer. Moncrief,[15] from the analysis of the results of many tests of columns, found the value

[15] Trans. Am. Soc. Civ. Engrs., Vol. 45, 1901, p. 334.

of $\frac{ec}{r^2}$ to range from 0.15 to 0.6, and he recommended the larger value to be on the safe side. He found that this value applies alike to cast iron, wrought iron, mild steel, high carbon steel and several kinds of timber.

TABLE 3

VALUES OF SEC $\frac{l}{2r}\sqrt{\frac{P}{aE}}$ IN SECANT FORMULA

Values of $\frac{P}{a}$	Values of $\frac{l}{r}$									
	50	60	70	80	90	100	110	120	130	140
2,000	1.02	1.03	1.04	1.05	1.07	1.09	1.11	1.13	1.16	1.19
5,000	1.05	1.08	1.11	1.15	1.20	1.25	1.32	1.40	1.50	1.62
8,000	1.09	1.14	1.19	1.26	1.35	1.46	1.60	1.79	2.05	2.41
10,000	1.12	1.17	1.25	1.34	1.47	1.63	1.86	2.18	2.65	3.46
14,000	1.17	1.25	1.37	1.54	1.77	2.12	2.68	3.69	6.02	16.4
18,000	1.22	1.35	1.53	1.79	2.21	2.95	4.51	9.86		
20,000	1.25	1.40	1.62	1.95	2.51	3.62	6.65	47.40		
22,000	1.28	1.45	1.71	2.13	2.90	4.65	12.3			
25,000	1.33	1.54	1.88	2.47	3.73	7.87				
28,000	1.38	1.64	2.08	2.93	5.15	22.90				
30,000	1.42	1.72	2.23	3.32	6.79					
32,000	1.46	1.79	2.41	3.83	9.86					
35,000	1.52	1.92	2.73	4.93	28.70					
38,000	1.59	2.07	3.13	6.80						
40,000	1.63	2.18	3.46	9.07						
42,000	1.68	2.31	3.87	13.30						
45,000	1.76	2.51	4.68	43.00						
48,000	1.85	2.76	5.88							
50,000	1.91	2.95	7.06							

Values of $sec\ \frac{l}{2r}\sqrt{\frac{P}{aE}}$

For a circular cross-section, if $\frac{ec}{r^2} = 0.6$, $e = 0.3r$ and hence for a column having a circular section 10 in. in diameter, e would equal $\frac{3}{4}$ in.

Prichard [16] proposes the following:

$$\frac{ec}{r^2} = \frac{1}{10} + \frac{1}{700}\cdot\frac{l}{r}. \quad \ldots\ldots\ldots \quad (169)$$

[16] Trans. Am. Soc. Civ. Engrs., Vol. 61, p. 173.

And when this is applied to ordinary structural-steel pin-ended columns in which the value of $\frac{c}{r}$ is usually about $\frac{7}{5}$,

$$e = 0.07r + 0.001\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

Thus a column having the relatively large $\frac{l}{r}$ of 100 ($r = 5$ and $l = 500$) the value of e is 0.85 in.

Basquin [17] proposes the following:

$$\frac{ec}{r^2} = 0.10 + 0.001\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

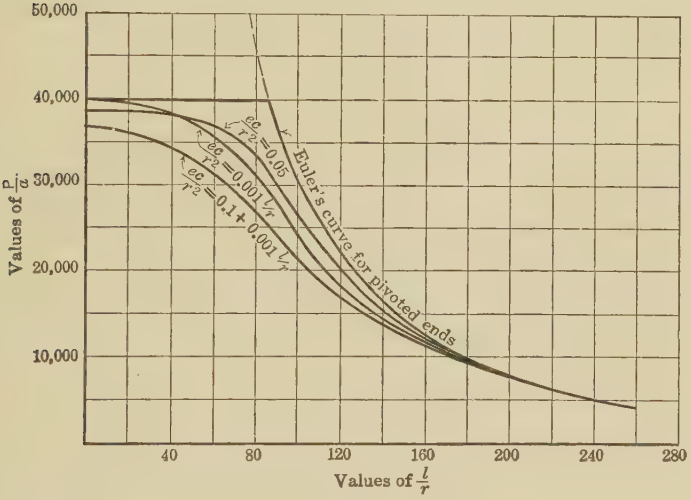

FIG. 212.—Effect of values of $\frac{ec}{r^2}$ on secant column formula.

In Fig. 212 are shown the curves representing the secant formula for steel columns, with different values of the eccentric ratio $\frac{ec}{r^2}$; the yield-point of the steel is assumed to be 40,000 lb. per sq. in.

[17] Journal Western Society Engrs., Vol. 18, 1913, p. 457. For a compilation of the recommendation of many authorities see Salmon's "Columns," Oxford Technical Publications, p. 148. This book also gives an excellent discussion of the factors entering in the various column formulas.

PROBLEMS

334. A column having the same cross-section as the column described in Prob. 299 (see Fig. 197) and a length of 25 ft. is subjected to an axial load of 120,000 lb. Find the maximum so-called unit-stress, assuming that $\frac{ec}{r^2} = 0.5$.

Ans. $s = 8530$ lb. per sq. in.

335. The column described in Prob. 318 is subjected to an axial load of 73,000 lb. Find the maximum value of s in the column; use eq. (169) for finding a value of $\frac{ec}{r^2}$.

Ans. $s = 9900$ lb. per sq. in.

336. Solve Prob. 313 by the secant formula using a value of 0.6 for $\frac{ec}{r^2}$.

337. Change the length of the column described in Prob. 312 to 6 ft. and solve by the secant formula using a value for $\frac{ec}{r^2}$ as given by eq. (171).

REVIEW QUESTIONS AND PROBLEMS

338. The following statements are not correct. Point out and correct the errors.

(*a*) The maximum axial load that can be resisted by a very slender column depends on the yield point of the material if the material is ductile, or on the ultimate compressive (or shearing) strength if the material is brittle.

(*b*) The maximum unit-stress in columns (particularly in built-up columns) having intermediate values of l/r is the sum of a direct stress and a bending stress, and therefore is given by the formula $s = \frac{P}{a} + \frac{Mc}{I}$ where M is the bending moment arising mainly from the deflection of the column.

(*c*) The end conditions of a column having an intermediate value of l/r affect the strength of the column more than do initial crookedness of the column, fortuitous eccentricity of loading, variation in the properties of the component parts in a built-up column, etc., and therefore the value of ϕ used in the Gordon-Rankine formula should change markedly with the end conditions.

339. Explain the apparent contradiction that the critical or buckling load given by Euler's formula for an axially loaded very slender column causes a stress in the column less than the proportional limit of the material, and yet a load only slightly greater than this critical value causes the column to collapse, that is, to fail completely.

340. The initial crookedness of columns is no greater, in general, than that of beams, and the action lines of the loads on columns deviate probably considerably less from the assumed action lines than do the loads on beams. But the strength of beams is affected very little by these conditions whereas the strength of columns (particularly columns of intermediate slenderness ratios) is influenced primarily by them. Explain.

341. In the formula $\frac{P}{a} = s$ for a short structural steel column subjected to an assumed axial load, why is the working value of s (12,500 lb. per sq. in.) taken so much lower than that (16,000 to 20,000 lb. per sq. in.) used for the same member if

subjected to an assumed axial *tensile* load? Also, what is the justification for advocating in recent years a decrease in the values of s and C in the straight-line formula $\frac{P}{a} = s - C\frac{l}{r}$? Is it because *bending* of medium-length columns is now considered of lesser importance and column crookedness and eccentricity of loading are considered of greater importance than they were formerly considered to be?

342. Do any of the commonly used column formulas give reliable results for a built-up column if the component parts of the column do not act as a unit? Explain.

343. One advantage that is sometimes claimed for the Gordon-Rankine formula (in contrast to the straight-line formula) is that it can be used for all values of l/r. If values of s and ϕ are chosen so that the formula gives reliable values of P/a for intermediate values of l/r, what can be said of the values of P/a obtained for large values of l/r?

344. What diameter should a circular structural steel bar have in order that it may carry an axial working load of 20,000 lb.? The bar is 5 ft. long and has pivoted ends. Assume that the Joint Committee straight-line formula is applicable, and after determining the diameter, calculate the l/r of the column to justify the use of the formula. *Ans.* $d = 1.76$ in.; $l/r = 136$.

345. Two very slender columns having the same lengths and cross-sections are subjected to axial loads. One column is made of steel and has round ends, and the other is made of aluminum and has fixed ends. What is the ratio of the maximum loads the two columns can carry? Consult Table 1 of Chapter I. *Ans.* $P_s = 0.75\ P_a$.

346. A steel column having a cross-section like that shown in Fig. 197 is 20 ft. long and has round ends. The column is subjected to a load P whose action line passes through a point on the XX axis 2 in. from the YY axis. Calculate a working value of P from equation 138(a), modified to take account of the eccentricity of loading. *Ans.* $P = 97{,}800$ lb.

CHAPTER IX

RELATIONS BETWEEN STRESSES AT A POINT ON DIFFERENT PLANES PASSING THROUGH THE POINT. THEORIES OF FAILURES

83. Introduction.—It is well to recall that a stress should not be thought of apart from the area on which it acts, and hence the idea of a unit-stress at a point, to be definite, must involve a plane passing through the point. The term " state of stress at a point " is sometimes used when the stresses on several planes passing through the point are involved.

Two important relations between the stresses at a point on different planes passing through the point have already been found. In Art. 27 it was shown that

> If a shearing unit-stress exists at a point on one plane in a stressed body there must occur an equal shearing unit-stress at the same point on a plane at right angles to the first plane.

And if on these two planes shearing stresses, only (no normal stresses), exist, the " state of stress at the point " is said to be one of *pure shear or simple shear.* Further, it was shown in Art. 28 that

> If a state of pure shear exists at a point in a body, normal stresses also exist at the same point, on planes bisecting the angles between the planes on which the shearing stresses occur, and these normal unit-stresses are equal in magnitude to the shearing unit-stresses.

Method of Attack.—Other states of stress at a point are considered in this chapter. It is important to observe that the method of analyzing the stresses in a member is the same whether the purpose is to obtain the relation between the external forces and the internal forces (stresses) or to obtain the relation between internal forces only. The method is to imagine a section to be passed through the member so that a portion of the member is severed from the member at the areas on which the stresses are to be found; the forces acting on this severed portion are then external to the portion, and since these forces must satisfy the conditions of equilibrium, the relations between them can

be found by applying the equations of equilibrium. If the imaginary section cuts the member completely in two, either one of the two parts will be held in equilibrium by the original external forces that act on that part and the stresses exerted by the other part at the cut section; a study of the equilibrium of all the forces acting on either one of these two parts will therefore give relations between the external forces or loads and the internal forces or stresses. By this procedure the equations $P = as$, $M = \frac{sI}{c}$, etc., were found in the preceding chapters. If, however, the portion cut from the body by the imaginary sections lies wholly within the body, the condition of equilibrium will give relations between stresses, only, because internal forces are the only forces, in addition to the weight, that act on the portion.

Now if the relations of the stresses *at a point* on different planes passing through the point are desired, the portion imagined to be severed from the body must be an indefinitely small part at the point in question. Therefore, the weight of the portion may be neglected, and further the forces on the boundary faces of the small portion may be assumed to be uniformly distributed on the faces. This method has already been used in Arts. 27 and 28.

Principal Stress Defined.—As a rule the stresses at a point that are required to be found are the maximum normal (tensile or compressive) unit-stress and the maximum shearing unit-stress. It will be shown in a later article that the maximum normal stress occurs on a plane on which the shearing unit-stress is zero. Any normal unit-stress at a point that occurs on a plane on which the shearing stress is zero is called a principal stress at the point. In general there are at any point in a stressed body three planes through the point mutually at right angles to each other on which no shearing stresses occur. There are, therefore, in general three principal stresses at any point in a stressed body; one of the principal stresses is the maximum normal stress at the point; one is the minimum normal stress and the other is, of course, intermediate between the other two. In the case of a plane state of stress one of the three principal stresses has the value of zero. Principal stresses produced by certain combinations of loading are found in the subsequent articles.

84. Maximum Shearing Stress in Terms of Principal Stresses.—*Proposition.*—The maximum value of the shearing unit-stress at a point in a stressed body is one-half of the algebraic difference of the maximum and minimum principal unit-stresses at the point; that is

$$Max.\ s_s = \tfrac{1}{2}\,(s_{\max.} - s_{\min.})$$

in which a compressive stress is to be considered to be a negative tensile stress. Further, this maximum shearing unit-stress occurs on each of the two planes that bisect the angles between (makes an angle of 45° with) the planes on which the maximum and minimum principal stresses occur.

Proof.—Let the block in Fig. 213 represent a small part of a stressed body subjected to tensile principal unit-stresses on the plane AB (and CD), a tensile principal unit-stress s_2 on the plane BD (and AC) and a principal stress of zero magnitude on the faces parallel to the plane of the paper.

If now an oblique plane is passed through the block and one part of the block is removed, the forces holding the remaining part in equilibrium are as shown in Fig. 213(*b*). By applying one of the equations of

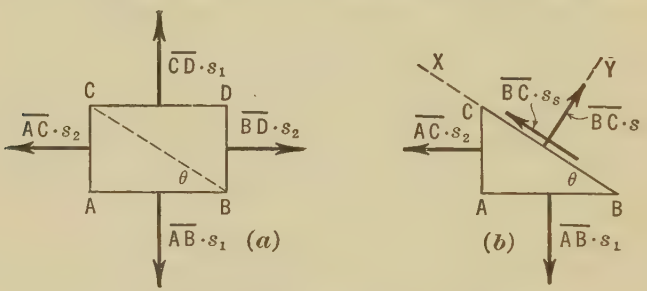

FIG. 213.—Shearing stress resulting from principal stresses.

equilibrium (resolving the forces parallel to the x-axis), the following equation is found:

$$\overline{BC}s_s = \overline{AB}s_1 \sin\theta - \overline{AC}s_2 \cos\theta.$$

But
$$\frac{AB}{BC} = \cos\theta; \quad \frac{AC}{BC} = \sin\theta.$$

Hence
$$s_s = (s_1 - s_2)\sin\theta\cos\theta = \tfrac{1}{2}(s_1 - s_2)\sin 2\theta. \quad . \quad . \quad (172)$$

But $\sin 2\theta$ is a maximum (equal to unity) when θ equals 45°, and $(s_1 - s_2)$ will be greatest when s_1 is the maximum of the three principal stresses at the point and s_2 is the minimum of the three. Hence

$$\text{Max. } s_s = \tfrac{1}{2}(s_{\text{max.}} - s_{\text{min.}}). \quad . \quad . \quad . \quad . \quad . \quad (173)$$

The maximum s_s will occur on planes that are perpendicular to the plane containing the vectors that represent the stresses $s_{\text{max.}}$ and $s_{\text{min.}}$ and that bisects the right angles between these vectors.

It is important to note that the parenthesis in equation (173) is the *algebraic difference* of the maximum and minimum principal stresses.

In Fig. 213(a) the maximum stress is $+s_1$ and the minimum stress is zero, and hence the maximum shearing unit-stress is $\frac{1}{2}s_1$ and occurs on a plane making 45° with the plane of the paper. Likewise if two of the principal stresses are zero as in a pure tension member, max. $s_s = \frac{1}{2}s_1$, as was shown also in Art. 4. If, however, s_2 in Fig. 213(a) were a compressive stress then $s_{\text{min.}}$ would be $(-s_2)$ and equation (173) would then become max. $s_s = \frac{1}{2}(s_1 + s_2)$, and this max s_s would occur on a plane perpendicular to the plane of the paper bisecting the angle between s_1 and s_2.

PROBLEMS

347. A boiler 6 ft. in diameter is made of plates $\frac{3}{4}$ in. thick and is subjected to an internal steam pressure of 200 lb. per sq. in. Find the maximum shearing unit-stress in the plate. On what planes does it act? Two of the principal stresses are shown in Fig. 214 and the third principal stress is 200 lb./in.² on the inner surface and zero on the outer surface. *Ans.* 4900 lb./in.²

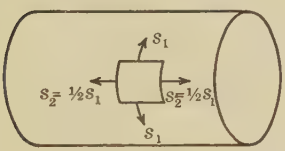

FIG. 214.

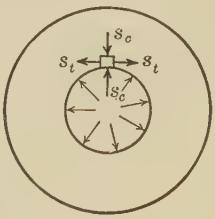

FIG. 215.—Stresses in thick cylinder.

348. A gun barrel or thick cylinder (Fig. 215) on which hoops are shrunk (not shown in Fig. 215) is subjected to an internal pressure of 50,000 lb. per sq. in. due to the explosion of the charge; the maximum radial compressive stress, s_c, in the material is also 50,000 lb. per sq. in. Now if the maximum circumferential stress, s_t, is 16,000 lb. per sq. in. and occurs at the same point in the cylinder as does s_c, find the maximum shearing unit-stress and indicate the planes on which it occurs. The third principal stress may be assumed to be zero.

349. At a certain point in a body two of the principal stresses have values of 2000 lb. per sq. in., one being a compressive stress and the other a tensile stress, the third principal stress being zero. Find the normal unit-stress and the shearing unit-stress on a plane making 30° with the direction of the principal tensile stress. *Ans.* $s_c = 1000$ lb. per sq. in. $s_s = 1730$ lb. per sq. in.

350. In Fig. 215 if $s_c = 16{,}000$ lb./in.², $s_t = 50{,}000$ lb./in.² and a third principal tensile stress of 60,000 lb./in.² acts normal to the plane of the paper, what is the maximum shearing unit-stress at the point, and on what planes does it act?

85. Maximum Normal Stress Occurs on Plane of Zero Shearing Stress. Ellipse of Stress.—In Art. 83, a principal stress was defined to be a normal stress that acts on a plane on which the shearing stress is zero. It will here be proved that a principal stress at a point in a

body, where stresses exist in one plane only (planar stress), is either the *maximum* or the *minimum* normal stress which occurs, at the point on any plane perpendicular to the plane of stress.

In Fig. 216, let ABC be an elementary prism at a point in any body; that is, AB, AC, and BC are three planes passing through a given point in the body. Let the dimension of the prism perpendicular to the paper be unity. Further, let the planes AB and AC be the planes on which principal stresses occur; the principal unit-stresses on these planes are denoted by s_1 and s_2 and are assumed to be tensile stresses. On any other plane, such as BC, there will occur both a shearing unit stress and a normal unit-stress; let the total or actual unit-stress on the plane BC be denoted by s_r, and let the coordinates of the extremity of the vector s_r be x and y.

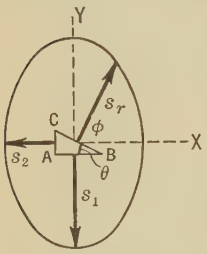

FIG. 216.—Ellipse of stress.

Now since the forces acting on the prism are in equilibrium, the sum of the x-components and of the y-components must equal zero. Hence,

$$(\overline{BC}s_r)\cos\phi = \overline{AC}s_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

and

$$(\overline{BC}s_r)\sin\phi = \overline{AB}s_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

but $s_r \cos\phi = x$, $s_r \sin\phi = y$, $\dfrac{AC}{BC} = \sin\theta$, and $\dfrac{AB}{BC} = \cos\theta$. Hence (174) and (175) may be written as follows:

$$x = s_2 \sin\theta \quad \text{or} \quad \frac{x}{s_2} = \sin\theta,$$

and

$$y = s_1 \cos\theta \quad \text{or} \quad \frac{y}{s_1} = \cos\theta$$

from which the following equation is obtained,

$$\frac{x^2}{{s_2}^2} + \frac{y^2}{{s_1}^2} = \sin^2\theta + \cos^2\theta = 1. \quad . \quad . \quad . \quad . \quad (176)$$

This is an equation of an ellipse having s_1 and s_2, the principal unit-stresses, as the semi-axes, and hence the locus of the extremity of the vector representing the unit-stress on the face BC as this face is assumed to turn through 360° is an ellipse. This ellipse is called the *ellipse of stress.*

Now, as may be seen from inspection, the unit-stress s_r has its maximum or minimum value when it is one of the semi-axes of the ellipse, that is, when it occurs on a plane on which there is no shearing stress.

86. Shearing Stresses Combined with One Normal Stress.—The problem discussed in this article may be stated as follows: Given, at a point in a body, a shearing unit-stress on each of two planes at right angles to each other, and a normal unit-stress on one of the two planes; required to find in terms of the given stresses the values of the resulting normal and shearing unit-stresses at the same point on other planes passing through the point, and in particular to find the maximum values of the stresses at the point.

Many structural and machine members are subjected to loads that cause the "given" stresses stated above, and these stresses may be found from the loads by use of the equations discussed in the preceding chapter. The "given" stresses may arise in the following ways:

(*a*) The shearing unit-stress, s_s, and the normal unit-stress, s, may be due to central loads, as in the bolt loaded as shown in Fig. 217, in which case s_s and s are found from the loads as follows:

$$s_s = \frac{Q}{a} \qquad \text{and} \qquad s = \frac{P}{a}. \quad . \quad . \quad . \quad . \quad . \quad (177)$$

(*b*) s_s may be due to torsional loads, and s to an axial load (Fig. 218), in which case

$$s_s = \frac{Tc}{J} \qquad \text{and} \qquad s = \frac{P}{a}. \quad . \quad . \quad . \quad . \quad . \quad (178)$$

(*c*) s_s may be due to torsional loads and s to bending loads, as in an engine crank shaft, etc., in which case

$$s_s = \frac{Tc}{J} \qquad \text{and} \qquad s = \frac{Mc}{I}. \quad . \quad . \quad . \quad . \quad . \quad (179)$$

(*d*) Both s_s and s may occur in a beam, in which case

$$s_s = \frac{V}{It}\int_{y_0}^{c} y da \quad \text{and} \quad s = \frac{Mc}{I}. \quad . \quad . \quad . \quad . \quad . \quad (180)$$

Further, as shown in Art 27, if a shearing unit-stress of intensity s_s occurs at a point on one plane, a shearing unit-stress equal to s_s must occur at the same point on a plane at right angles to the first.

Fig. 217 shows a bolt loaded, as suggested under (*a*) above, so that there is produced the combination of stresses under discussion; the

stresses are shown in Fig. 217 acting on the elementary block in the bolt, and Fig. 218 shows a similar block subjected to the same combination of stresses by the method of loading suggested under (*b*) above. Fig. 219(*a*) shows one of these elementary blocks, enlarged, and also the forces that act on its four faces and that hold it in equilibrium;

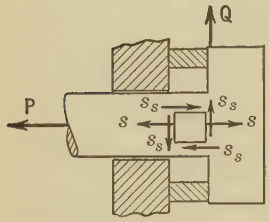

FIG. 217.—Combined shearing and tensile stresses.

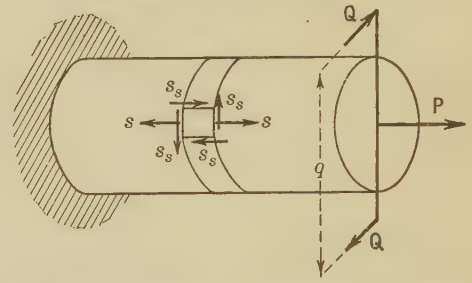

FIG. 218.—Combined shearing and tensile stresses.

the dimension perpendicular to the plane of the paper is assumed for convenience to be unity.

To Find the Maximum Normal Stress.—Let a plane be passed through the elementary block of Fig. 219(*a*) at an angle θ with the plane on which the normal stress occurs and perpendicular to the plane of the paper. The normal and shearing unit-stresses on this inclined plane will be denoted by s' and s'_s, respectively, the normal

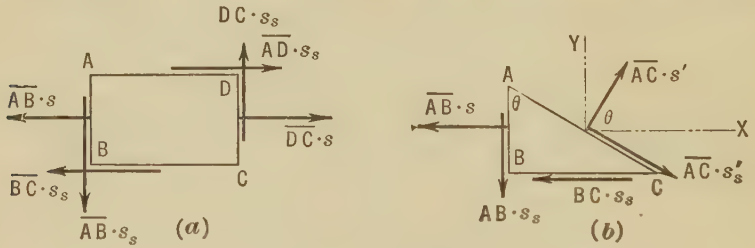

FIG. 219.—Combined shearing and normal stresses.

stress here being a tensile stress. The forces, then, that hold the block ABC in equilibrium are as indicated in Fig. 219(*b*). And since to satisfy the conditions of equilibrium the sum of the components of the forces in the x-direction and in the y-direction must each be equal to zero, we may write,

$$\overline{AC}s' \cos\theta + \overline{AC}s'_s \sin\theta - \overline{BC}s_s - \overline{AB}s = 0 \quad . \quad . \quad (181)$$

and

$$\overline{AC}s' \sin\theta - \overline{AC}s'_s \cos\theta - \overline{AB}s_s = 0. \quad . \quad . \quad (182)$$

By dividing each of these equations by $\overline{AC}$ and noting that $\frac{BC}{AC} = \sin\theta$ and $\frac{AB}{AC} = \cos\theta$, the equations obtained are:

From (181),

$$(s' - s)\cos\theta + (s'_s - s_s)\sin\theta = 0. \quad . \quad . \quad . \quad . \quad (183)$$

From (182),

$$s'\sin\theta - (s'_s + s_s)\cos\theta = 0. \quad . \quad . \quad . \quad . \quad (184)$$

The stresses s' and s'_s on any plane making an angle θ with the plane on which the normal stress s occurs may be found by use of these equations if values of s and s_s are known.

Now, as shown in Art. 85, *the normal unit-stress, s', has its maximum or minimum value on the plane on which the shearing unit-stress, s'_s,* is zero. Therefore, if s'_s in the above equations is made equal to zero, and if θ' is used to denote the corresponding value of θ, the above equations become,

$$(s' - s)\cos\theta' - s_s\sin\theta' = 0. \quad . \quad . \quad . \quad . \quad (185)$$

$$s'\sin\theta' - s_s\cos\theta' = 0, \quad . \quad . \quad . \quad . \quad (186)$$

in which s' is the maximum or minimum normal stress that occurs at the point; that is, s' is a principal stress. By eliminating θ' from these two equations, the maximum and minimum values of s' may be found in terms of s and s_s, and these in turn may be found from the external loads by the methods already discussed. Likewise, by eliminating s' from the two equations the values of θ' may be found which give the directions of the planes on which the principal stresses occur.

By dividing (185) by (186) the following equation is obtained:

$$\frac{s' - s}{s_s} = \frac{s_s}{s'} \quad \text{or} \quad s'^2 - ss' - s_s^2 = 0.$$

Therefore,

$$\left.\begin{aligned} s' &= \tfrac{1}{2}s \pm \sqrt{\left(\frac{s}{2}\right)^2 + s_s^2} \\ \text{or,} \qquad s' &= \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{s^2 + 4s_s^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (187)$$

Thus, there are two principal unit-stresses, one being the *maximum*

normal unit-stress (in this case a tensile stress) given by the expression

$$\text{max. } s' = \tfrac{1}{2}s + \sqrt{\left(\frac{s}{2}\right)^2 + s_s^2} \quad . \quad . \quad . \quad . \quad (188)$$

and the other the minimum normal unit-stress given by the expression

$$\text{min. } s' = \tfrac{1}{2}s - \sqrt{\left(\frac{s}{2}\right)^2 + s_s^2}. \quad . \quad . \quad . \quad . \quad (189)$$

But the last term of the right side of this equation is always greater than $\frac{1}{2}s$, and hence min. s' is a negative tensile stress, that is, a compressive stress, and may be written

$$-\text{min. } s' = s'_c = -\tfrac{1}{2}s + \sqrt{\left(\frac{s}{2}\right)^2 + s_s^2}. \quad . \quad (190)$$

Further, the planes on which these principal unit-stresses occur are at right angles to each other, as is proved in the next paragraph.

In eq. (188) max. s' will be a tensile stress if s is a tensile stress, and min. s' (eq. 189) will be a compressive stress if s is a tensile stress. Likewise, in eq. (188) max. s' will be a compressive stress if s is a compressive stress, and min. s' (eq. 189) will be a tensile stress if s is a compressive stress.

Direction of Planes on which Principal Stresses Occur.—The value of θ' in eqs. (185) and (186) may be found by eliminating s' from these equations:

From (186)

$$s' = s_s \frac{\cos \theta'}{\sin \theta'}.$$

Substituting this value of s' in (185) we have:

$$s_s \frac{\cos^2 \theta'}{\sin \theta'} - s \cos \theta' = s_s \sin \theta'$$

or

$$s_s(\cos^2 \theta' - \sin^2 \theta') = s \sin \theta' \cos \theta'$$

whence

$$s_s \cos 2\theta' = \tfrac{1}{2}s \sin 2\theta'$$

Therefore,

$$\tan 2\theta' = \frac{2s_s}{s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (191)$$

Now there are always two angles, between 0° and 360°, for which the tangents are equal, the two angles differing by 180°. Thus, there are two values of $2\theta'$ that differ by 180°, and hence there are two values of θ' that differ by 90°, as indicated in Fig. 220. Therefore, *the principal stresses occur on planes that are perpendicular to each other*, and the direction of the principal planes may be found by determining the value of θ' from eq. (191); θ' is measured from the plane on which the normal stress, s, occurs. The position of an elementary block on which the principal stresses act, as they occur in the bolt of Fig. 217, is shown in Fig. 221(*b*).

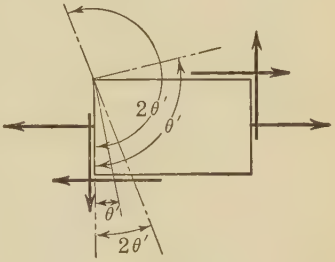

FIG. 220.

It will be noted that θ' is measured from the plane on which the normal stress acts, and that it will be measured in a clockwise direction or a counter-clockwise direction according to the sign of s (whether a tensile or compressive stress) and according to the direction of s_s (whether it acts up or down). The direction in which θ' should be measured, however, may as a rule be determined easily by inspection, by noting

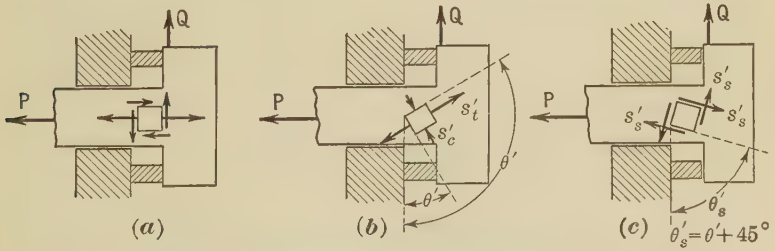

FIG. 221.—Principal stresses and maximum shearing stresses in bolt.

that, if the shearing stresses s_s were the only stresses acting, the maximum principal stress would be on one of the 45° planes, which may be easily determined by inspection in any case, and if s were the only stress acting it would be the maximum principal stress and hence θ' would be equal to zero. Therefore, if both s_s and s act, the plane of the maximum principal stress will be between the plane on which s acts and the 45° plane; these facts are indicated in Fig. 222.

The Maximum Value of the Shearing Unit-Stress s'_s.—As shown in Art. 84, the maximum shearing unit-stress resulting from two prin-

cipal stresses is one-half the algebraic difference of the principal stresses. Therefore, referring to Fig. 221, we have

$$\text{max. } s'_s = \tfrac{1}{2}(\text{max. } s' - \text{min. } s') = \tfrac{1}{2}(s'_t + s'_c)$$

$$= \tfrac{1}{2}\left[\left(\tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}\right) + \left(-\tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}\right).\right]$$

Hence,

$$\text{max. } s'_s = \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2}$$

or

$$\text{max. } s'_s = \sqrt{\left(\frac{s_t}{2}\right)^2 + s_s^2} \qquad \text{. . . . . (192)}$$

And, as shown in Art. 84, this maximum shearing unit-stress occurs on each of two planes that make angles of 45° with each of the planes on which the principal stresses occur. That is, *the planes of maximum shear bisect the angles between the planes on which the principal stresses occur*. The planes on which the maximum shearing stresses occur are shown in Fig. 221(*c*), in which $\theta_s = \theta' + 45°$. There are normal stresses also on the planes on which the maximum shearing stresses occur and these normal stresses are of equal magnitude (these normal stresses are not shown in Fig. 221*c*).

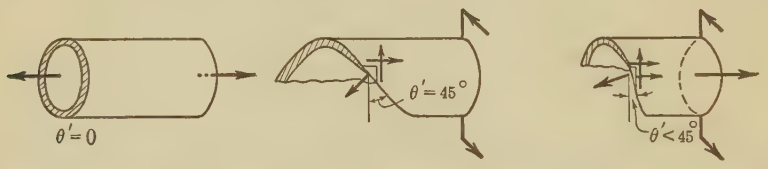

FIG. 222.—Directions of planes on which principal stresses act shown by areas of rupture of brittle material.

The student should draw diagrams similar to those in Fig. 221 for a shaft loaded as indicated in Fig. 218, and explain why the cast-iron specimens shown in Fig. 222 failed as indicated. Since cast iron is weak in tension, failure in each case is by tension.

ILLUSTRATIVE PROBLEMS

Problem 351.—If in Fig. 223 the axial tensile load, P, is equal to 45,000 lb., the twisting moment, Qq, is equal to 30,000 lb.-in. and the diameter, d, is equal to 3 in., find the maximum normal and shearing unit-stresses and the directions of the planes on which these stresses occur.

Solution.—The unit-stress, s_t, caused by the load P is

$$s_t = \frac{P}{a} = \frac{45{,}000}{\frac{\pi(3)^2}{4}} = 6360 \text{ lb. per sq. in.}$$

and the shearing unit-stress, s_s, caused by the torsional moment T or Qq is

$$s_s = \frac{Tc}{J} = \frac{30{,}000 \times 1.5}{\frac{\pi(3)^4}{32}} = 5660 \text{ lb. per sq. in.}$$

The planes on which these stresses occur at an outer fiber of the shaft, are shown in Fig. 223(*a*).

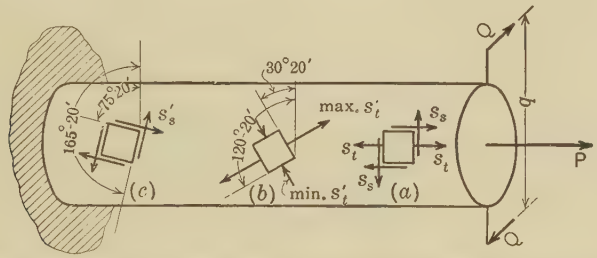

FIG. 223.—Stresses in bar subjected to combined tensile and torsional loads.

The principal stresses, s'_t and s'_c, and the maximum shearing unit-stresses, s'_s, resulting from the above stresses are,

$$\text{max. } s'_t = \tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2} = \tfrac{1}{2}\cdot 6360 + \tfrac{1}{2}\sqrt{(6360)^2 + 4(5660)^2}$$

$$= 3180 + 6480 = 9660 \text{ lb. per sq. in., tensile stress.}$$

$$\text{min. } s'_t = \tfrac{1}{2}s_t - \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2} = -3300 \text{ lb. per sq. in., compressive stress,}$$

$$s'_s = \pm\tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2} = 6480 \text{ lb. per sq. in.}$$

and the planes on which the principal stresses occur make the angles θ' with the plane on which the stress s_t occurs, the values of θ' being found as follows:

$$\tan 2\theta' = \frac{2s_s}{s_t} = \frac{2 \times 5660}{6360} = 1.78.$$

Hence

$$2\theta' = 60° 40' \text{ or } 240° 40'.$$

and

$$\theta' = 30° 20' \text{ or } 120° 20'$$

as indicated in Fig. 223(*b*). Further, the angles θ'_s that the planes of maximum shear make with the plane on which s_t occurs are

$$\theta'_s = \theta' + 45° = 75° 20' \text{ and } 165° 20'.$$

These planes are shown in Fig. 223(*c*).

Problem 352.—A pressure P of 10,000 lb. on the crank pin of the steel crank-shaft shown in Fig. 224(*a*) is required to turn the shaft at constant speed when the shaft is subjected to a constant resisting torque. If the diameter, d, of the shaft is 4 in., find the maximum combined normal and shearing unit-stresses at the section AB. Also find the ratios of the maximum normal and shearing stresses to the corresponding yield-points of the material if the tensile yield-point is 40,000 lb. per sq. in. P acts perpendicular to the pin and to the crank.

Solution.—By considering a free body diagram of the part of the shaft to the right of section AB, it is evident that the only external force acting on this part is P, and hence the stresses at section AB are caused by P and must be such as to hold P in equilibrium. Now by introducing two equal, opposite, and collinear forces P_2 and P_3 at H (Fig. 224*b*), P is resolved into a bending load P_3 and a torsional couple P, P_2; and these three forces cause the same stresses on the section AB as does the original single force P. Thus the shaft to the right of the section AB is subjected to combined bending and torsional loads.

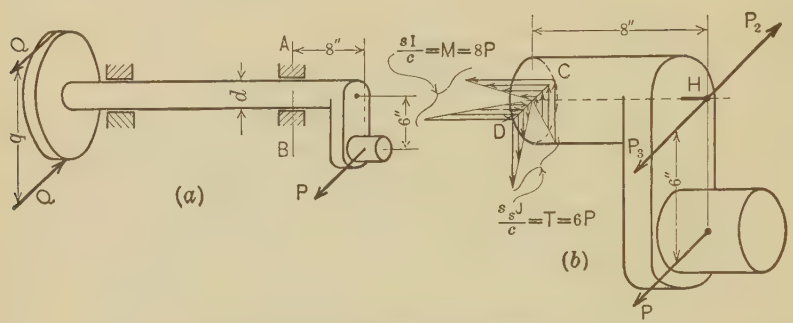

FIG. 224.—Stresses in crankshaft.

The bending moment at section AB due to the load P_3 is $8P$, and is held in equilibrium by the resisting moment $\frac{sI}{c}$ on the section AB as shown in Fig. 224(*b*). Thus the compressive unit-stress s at D (or the tensile unit-stress at C) is

$$s = \frac{Mc}{I} = \frac{8 \times 10{,}000 \times 2}{\frac{\pi(4)^4}{64}} = 12{,}730 \text{ lb. per sq. in.}$$

The twisting moment due to the forces P and P_2 is $6P$ and is held in equilibrium by the shearing resisting moment $\frac{s_s J}{c}$ on section AB as shown in Fig. 224(*b*). Thus the shearing unit-stress s_s at D (or at C) is

$$s_s = \frac{Tc}{J} = \frac{6 \times 10{,}000 \times 2}{\frac{\pi(4)^4}{32}} = 4770 \text{ lb. per sq. in.}$$

The maximum combined normal unit-stress at C and at D, then, is

$$s' = \tfrac{1}{2}s + \tfrac{1}{2}\sqrt{s^2 + 4s_s^2} = \tfrac{1}{2} \times 12{,}730 + \tfrac{1}{2}\sqrt{(12{,}730)^2 + 4(4770)^2}$$

$$= 6365 + 7935 = 14{,}300 \text{ lb. per sq. in.}$$

This is a compressive stress at D and a tensile stress at C. The maximum combined shearing unit-stress at C and at D is

$$s'_s = \tfrac{1}{2}\sqrt{s^2 + 4s_s^2} = 7935 \text{ lb. per sq. in.}$$

If the tensile yield-point of the steel is 40,000 lb. per sq. in. and the shearing yield-point is six-tenths of the tensile yield-point (as found from tests), then the maximum tensile stress is approximately 0.36 of the tensile yield-point, and the maximum shearing stress is 0.33 of the shearing yield-point.

PROBLEMS

353. If the values of P and Q in Fig. 221(a) are 8000 lb. and 6000 lb. respectively, and the diameter of the bolt is $\frac{3}{4}$ in., what are the maximum normal (tensile) and shearing unit-stresses developed in the bolt? If the bolt is made of steel having a tensile yield-point of 45,000 lb. per sq. in. and a shearing yield-point equal to 0.6 of the tensile yield-point, what are the factors of safety in tension and shear?

Ans. 1.78; 1.66.

354. A steel bar having a diameter d of 2 in. is held at one end (Fig. 225) and has a wrench indicated by AB attached at the other end; a force P of 300 lb. is applied to the end of the wrench handle as indicated in Fig. 225. Calculate the maximum normal and shearing unit-stresses developed in the bar.

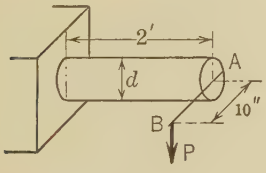

FIG. 225.

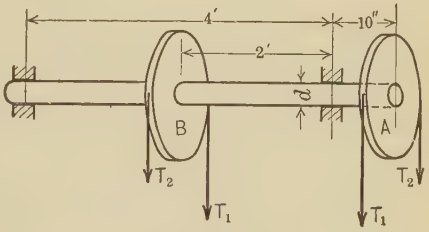

FIG. 226.

355. Two pulleys, A and B, are mounted on a shaft as shown in Fig. 226. The driving pulley, A, transmits 10 h.p. to the shaft and the driven pulley B. It turns the shaft at 100 r.p.m. The smaller belt tension, T_2, is 320 lb. and the diameter, d, of the shaft is 2 in. and the diameter of each pulley is 2 ft. Find the maximum normal and shearing unit-stresses in the shaft.

Ans. s'_t = 15,800 lb. per sq. in.; s'_s = 8400 lb. per sq. in.

356. The crank-pin pressure, P (Fig. 227b), perpendicular to the crank is 5000 lb. when the shaft is turning against a constant resisting moment Qq. The diameter of the shaft at A, B, and C is 3 in. Find the reactions R_1 and R_2 of the bearings and also the maximum normal and shearing unit-stresses at B.

Ans. s' = 8500 lb./in.^{2}; s'_s = 4720 lb./in.^{2}

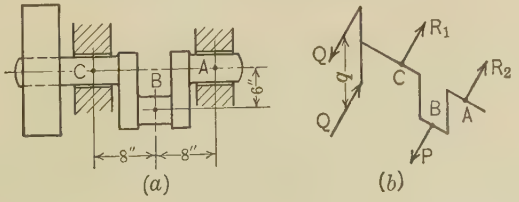

FIG. 227.

357. A solid steel propeller shaft 6 in. in diameter transmits 400 horsepower at a speed of 120 r.p.m. If the thrust of the propeller is 80,000 lb., what is the maximum

principal unit-stress and maximum shearing unit-stress in the shaft, assuming that the bending of the shaft is negligible.

358. In Fig. 228 a shaft 2 in. in diameter is supported in flexible bearings at A and D, and two gears B and C are keyed to the shaft. The gears are acted on by tangental forces as shown. Calculate the maximum principal tensile unit-stress and the maximum shearing unit-stress and show in a sketch approximately where in the shaft they occur.

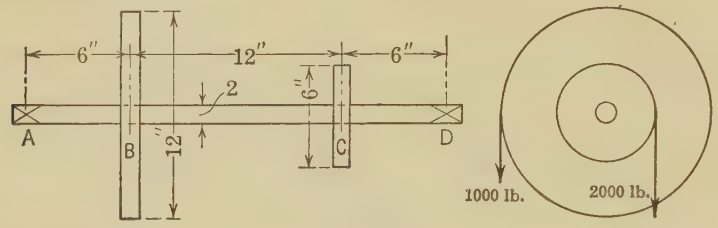

FIG. 228.

359. In Fig. 225 assume that the bar is hollow and that the inside diameter $d_1 = \frac{3}{4}d$, where d is the outside diameter. Find the value of d if the maximum shearing unit-stress in the bar is 8000 lb./in.^{2} The value of P is 400 lb.

Ans. $d = 2.13$ in.

360. In Fig. 228 let the 2000-lb. force act at the top of the gear C in the horizontal direction to the right instead of acting vertically downward as shown in the figure, and calculate the maximum principal and shearing unit-stresses in the shaft.

361. A steel shaft 4 in. in diameter and 10 ft. long resists a twisting moment. The total angle of twist is 0.10 radian. Find the maximum principal unit-stresses and show by a sketch where in the shaft they act. It is assumed that the proportional limit of the material is not exceeded.

362. If a shaft is subjected to combined torsion and bending, and the shearing yield-point of the material is six-tenths of the tensile yield-point, prove that the ratio of the shearing unit-stress s'_s to the shearing yield-point will be equal to the ratio of the tensile unit-stress s'_t to the tensile yield-point when the ratio of the bending moment M to the twisting moment T is equal approximately to 0.9.

363. A shaft 4 in. in diameter is subjected to an axial end thrust of 12 tons, and also a bending moment of 8000 lb.-ft. and a twisting moment of 10,000 lb.-ft. Find the maximum normal and shearing unit-stresses in the shaft.

Ans. $s' = 21{,}500$ lb. per sq. in. $s'_s = 12{,}900$ lb. per sq. in.

364. A shaft 5 in. in diameter resists a bending moment of 120 ton-in. and a twisting moment of 70 ton-in. Find the magnitude of the maximum normal and shearing unit-stresses.

Ans. $s'_t = s'_c = 10.6$ ton per sq. in.; $s'_s = 5.71$ ton per sq. in.

365. In which of the two shafts described below is the greater normal unit-stress developed and in which is the greater shearing unit-stress developed? Calculate the maximum unit-stresses in each shaft.

(1) A 4-in. shaft subjected to a twisting moment of 40 ton-in. and a bending moment of 32 ton-in.

(2) A 2-in. shaft subjected to a twisting moment of 7.0 ton-in. and a bending moment of 3 ton-in.

87. Diagonal Tensile Stress in a Beam.—The normal unit-stress on any right cross-section of a beam such as section AA, Fig. 229, is the greatest at the outermost fiber and is found from the flexure formula $s = \frac{Mc}{I}$, where c is the distance from the neutral axis to the most remote fiber and M is the bending moment at the section considered. And since the shearing stress at the outermost fiber is zero (Art. 38), the stress s is the maximum normal stress at this point of the beam as is indicated at the outer fibers in section AA' of Fig. 229.

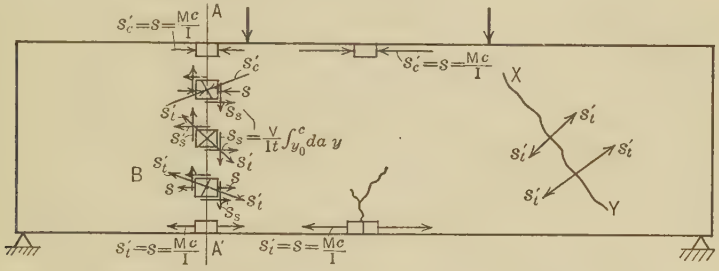

FIG. 229.—Resultant stresses in one section of beam.

Now in most beams the diagonal tensile unit-stress at any point in a cross-section is less than the tensile stress at the outermost fiber, and hence need not be found. In reinforced concrete beams, however, the diagonal tensile stress is often of importance since concrete is very weak in tension and, if the beam has only longitudinal reinforcing bars in the lower part of the beam, the diagonal tension may cause cracking of the beam in that portion of the beam where the shearing stresses are relatively large; the diagonal tension crack for the beam loaded as shown in Fig. 229 would occur along a line similar to XY.

The relative directions of the maximum normal stresses at various points along a section AA' are shown in Fig. 229, and the relation of these to the diagonal tension crack XY is also shown. In the central portion of the beam where no shear exists the cracks are approximately vertical as indicated in Fig. 229.

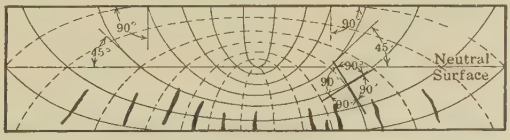

FIG. 230.—Directions of resultant stresses in beam.

The manner in which the directions of the principal stresses vary in a simple beam are shown in Fig. 230; the full lines refer to tensile stresses and the dotted lines to compressive stresses.

In order to prevent failure of concrete beams by diagonal tension, some of the longitudinal bars are usually bent up at the ends (see Art. 162) so that the bars have approximately the same direction as the diagonal tensile stresses.

A diagonal tension failure is frequently called a shear failure and the diagonal reinforcing bars are frequently called shear bars, but it should be kept clearly in mind that the failure is due to tensile stresses and the bars are stressed in tension.

PROBLEMS

366. The load acting on the short cantilever beam (Fig. 231) is 800 lb. (*a*) Find the unit-stress at point *A* normal to the vertical section. (*b*) Find the horizontal and vertical shearing unit-stresses at *A*. (*c*) Find the principal unit-stresses at *A*.

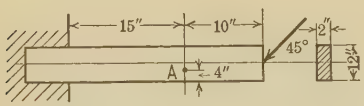

FIG. 231.

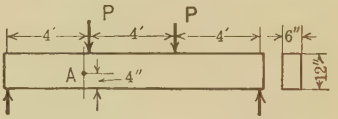

FIG. 232.

367. The loads *P* acting on the simple beam shown in Fig. 232 are each equal to 3800 lb. Find the diagonal tensile unit-stress at the point *A*.

Ans. $s' = 433$ lb./in.^{2}

88. Strain Due to Principal Stresses. Expression for $E\epsilon$.—The maximum unit-strain at a point in a body may be expressed in terms of the principal unit-stresses at the point (assuming stresses in one plane, that is, assuming the third principal stress to be zero) as follows.

Let Fig. 233 represent the point at which are acting the principal tensile unit-stress s_1 and the principal compressive unit-stress s_2. The unit-strain in the direction of s_1 caused by s_1 is

$$\epsilon_1 = \frac{s_1}{E},$$

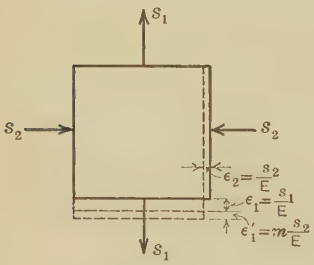

FIG. 233.—Strain due to two principal stresses.

and the unit-strain, ϵ_2, in the direction of s_2 caused by s_2 is

$$\epsilon_2 = \frac{s_2}{E},$$

and the unit-strain ϵ'_1 in the direction of s_1 caused by s_2 is

$$\epsilon'_1 = m\epsilon_2 = m\frac{s_2}{E},$$

in which m is Poisson's ratio (see Art. 6), and the modulus of elasticity, E, is assumed to be the same in compression as in tension. Therefore, the total unit-strain in the direction of s_1 is

$$\epsilon = \frac{s_1}{E} + m\frac{s_2}{E},$$

or

$$E\epsilon = s_1 + ms_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (193)$$

And if both stresses are tensile stresses $E\epsilon = s_1 - ms_2$.

Now if s_1 and s_2 are caused by one of the combinations of loading discussed in Art. 86 which produces shearing unit-stresses s_s on each of two planes and a normal unit-stress s on one of the two planes as illustrated in Fig. 234, the value of $E\epsilon$ may be expressed in terms of s and s_s as follows:

$$E\epsilon = s'_t + ms'_c$$

and making use of Eqs. (188) and (190) we have,

$$E\epsilon = \tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2} + m(-\tfrac{1}{2}s_t + \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2})$$

$$= \tfrac{1}{2}(1 - m)s_t + \tfrac{1}{2}(1 + m)\sqrt{s_t^2 + 4s_s^2}. \quad . \quad . \quad . \quad . \quad (194)$$

For steel the value of m found from experimental results is 0.25 to 0.30. Thus, for steel, the above equation becomes,

$$E\epsilon = \tfrac{3}{8}s_t + \tfrac{5}{8}\sqrt{s_t^2 + 4s_s^2}, \quad \text{when} \quad m = 0.25, \quad . \quad . \quad . \quad (195)$$

and

$$E\epsilon = 0.35s_t + 0.65\sqrt{s_t^2 + 4s_s^2}, \quad \text{when} \quad m = 0.30. \quad . \quad (196)$$

Now $E\epsilon$ may be called the *equivalent uni-directional unit-stress*, since, if ϵ were caused by a load that developed a stress in one direction

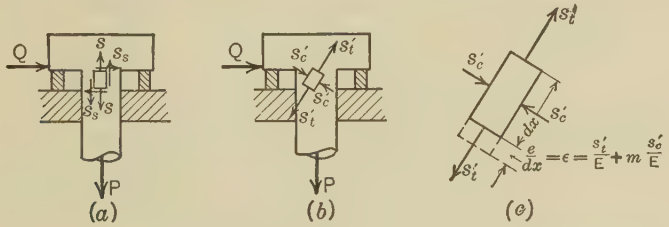

FIG. 234.—Strain due to principal stresses.

only (uni-directional stress), the value of $E\epsilon$ would then be numerically equal to the actual unit-stress. The significance and use of the quantity $E\epsilon$ is discussed in the next article.

PROBLEM

368. If the bar shown in Fig. 223 (refer to Prob. 351 for data) is made of steel for which Poisson's ratio is 0.25, what axial load P alone (with no twisting moment acting) could be applied to a similar bar of the same material and dimensions so that the maximum unit-strain in the two bars would be the same? *Ans.* 74,100 lb.

89. Theories of Failure of Elastic Action and Their Application.— Several modes of failure of members in engineering structures and machines were discussed briefly in Art. 11. It was there pointed out that, in general, members fail either by yielding (excessive permanent deformation) or by fracture (breaking). If failure occurs by yielding, the material is usually classified as ductile, and if by fracture it is classified as brittle, although there is no sharp line of demarkation between ductile and brittle materials.

Most of our structural metals fail by yielding when subjected to static (steady, or gradually applied) loads at ordinary temperatures, and it is of importance to determine, if possible, the mechanical condition in the metal that causes this yielding or plastic deformation to begin. Three of the more common theories of the cause of the breakdown of elastic action will here be considered briefly.

If a bar is subjected to a gradually increasing axial tensile load *in one direction only*, the material, when the load reaches a certain value, will begin to acquire permanent deformation and hence to fail in elastic action, and the reason for the failure may be (1) because the normal (tensile) unit-stress $\left(s_t = \frac{P}{a}\right)$ reaches the tensile elastic limit, s_e (or proportional limit), or (2) because the shearing unit-stress reaches the shearing elastic limit; this requires that the shearing elastic limit shall be equal to or less than one-half the tensile elastic limit, since, as shown in Art. 4, the maximum shearing unit-stress, on a 45-degree oblique plane, is one-half the tensile unit-stress, or (3) because the unit-deformation reaches a value which cannot be exceeded without having part of the deformation plastic and hence permanent. It is impossible to determine from a simple tension test, that is, from a test in which the stress is in one direction only, what is the real cause of the beginning of inelastic action in a material, because the above limiting values occur simultaneously. The three main theories suggested above may be stated briefly as follows:

1. The *maximum normal stress* theory, often called *Rankine's* theory, states that elastic action at any point in a material ceases, or inelastic action begins, *only* when the maximum normal (tensile) stress on a certain plane passing through the point reaches a value equal

to the tensile elastic limit found in a simple tension test, regardless of the normal or shearing stresses that occur on other planes through the point. Thus, if the block in Fig. 235(a) reaches its elastic limit when subjected to the stress s_1, the elastic limit will still be s_1 even if the block is subjected to the stress s_2 (Fig. 235b) in addition to s_1.

2. The *maximum strain* theory, often called *Saint Venant's* theory, states that inelastic action at any point in a body due to any combination of stresses at the point begins *only* when the maximum unit-elongation ϵ at the point reaches a value equal to that which occurs when inelastic action begins in a bar subjected to an axial tension test; which is the value of ϵ that occurs simultaneously with the tensile elastic limit s_e of the material. Now since $s_e = E\epsilon$ when the stress is applied in one direction only, the maximum strain theory states that inelastic action begins at any point in a body when the value of $E\epsilon$ at that point reaches a value equal to the tensile elastic limit of the material.

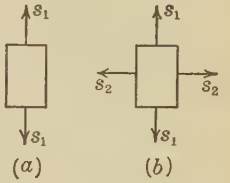

FIG. 235.

For example, according to this theory inelastic action in the block of Fig. 235(a) begins when s_1 equals the elastic limit of the material, but inelastic action in the block of Fig. 235(b) does not begin until $E\epsilon$ in the direction of s_1 is equal to the elastic limit of the material, and this value of $E\epsilon$ will not occur until s_1 in Fig. 235(b) is larger than s_1 of Fig. 235(a). For, the unit-deformation in the direction s_1 of the block in Fig. 235(a) is made less by the amount $m\frac{s_2}{E}$ when the stress s_2 is applied as in Fig. 235(b) (see Art. 88), and hence the stress s_1 must be increased before the limiting unit-deformation is reached.

3. The *maximum shearing stress* theory, sometimes called Guest's theory, states that inelastic action at any point in a body begins *only* when the maximum shearing unit-stress on some plane through the point reaches a value equal to the shearing elastic limit found by testing the material in simple shear as in a torsion test. And, as noted above, this theory when applied to a simple tension test of a bar requires that the shearing elastic limit shall be not greater than one-half the tensile elastic limit.

Other Theories.—Two theories of failure involving strain energy have been proposed, one by Beltrami and by Haigh and the other by von Mises and others. This latter theory requires that the shearing elastic limit of the material shall be $\frac{1}{\sqrt{3}} = 0.577$ times the tensile elastic limit,

which seems to check test results for steel rather closely. For further discussion of this topic see "Plasticity" by A. Nádai, Chap. 13, McGraw-Hill Book Co.

Significance of the Theories.—It must be kept clearly in mind that the theories of failure do not attempt to state what stresses are developed in a body. The only uncertainty which the theories attempt to explain is the state or condition developed in the material that causes inelastic action in the material to start.

Several experimental investigations have been carried out on ductile material, such as steel and brass, with combinations of stresses designed to make clear the cause of inelastic action, and although the results are not entirely concordant, the best conclusion appears to be that in *ductile* metals inelastic action begins when the shearing unit-stress reaches the shearing elastic limit of the material unless the limiting value of $E\epsilon$ is reached before the shearing elastic limit is reached, in which case inelastic action begins when the limiting value of $E\epsilon$ is developed. The tests indicated, however, that the shearing elastic limit of commercial hot-rolled steel is somewhat greater than one-half of the tensile elastic limit as demanded by the maximum shear theory, the value of the shearing elastic limit for such steel being close to six-tenths (0.6) of the tensile elastic limit. The maximum energy theory also agrees reasonably well with test results for the combinations of stress used in the tests, but it is rather involved and complicated to apply.

In many problems all these theories lead to the same results. In some problems the difference in the results obtained by the four theories are not important, and in some problems the assumptions made as to the distribution of loads and stresses and the assumption of homogeneity of material contain error that would not justify strong reliance on the results from one theory in preference to those from another. Partly for these reasons and partly owing to long usage and to the too frequent practice of assuming that the low value of the working stress is sufficient to compensate for all uncertainties, the maximum stress theory has been most commonly used, although the maximum shear theory, for ductile material, has gained rather wide acceptance in recent years. The maximum strain theory has been used extensively in the design of guns. The strain-energy theory is the least convenient to apply and has been used very little in design.

The sensible course to follow would seem to be to design a member so that the maximum shearing stress, the maximum principal stress and maximum principal strain do not exceed, respectively, some proportion of the shearing elastic limit, tensile (or compressive) elastic

limit and strain corresponding to the tensile elastic limit. This procedure is indicated in the illustrative problem which follows. In any case, the theories do not take account of all the conditions that the engineer must consider in the problem of failure, even of failure due to static loads. At best, the theories can be true only in a statistical sense.

Application of Theories of Failure of Elastic Action.—As stated above, the dimensions that should be assigned to a member that is to be subjected to loads causing combined stresses depend on the theory held concerning the cause of the breakdown of elastic action. This fact will be illustrated in the solution of a problem of the design of a member subjected to loads that cause the combination of stresses discussed in Art. 86, namely, shearing stresses on two planes at right angles to each other and a normal stress on one of the planes. In the following problem, this combination of stresses is caused by combined bending and torsion.

ILLUSTRATIVE PROBLEM

Problem 369.—The crank-shaft shown in Fig. 224 is made of steel having a tensile and compressive proportional limit of 40,000 lb. per sq. in. Determine, by each of the three theories of failure, the diameter, d, the shaft should have to resist the load P without developing more than one-half of the maximum elastic strength of the material. Tests show that the shearing proportional limit of steel is approximately 0.6 of the tensile proportional limit.

Solution.—As noted in Problem 352, the twisting moment T and the bending moment M have the following values:

$$T = 6P = 60{,}000 \text{ lb.-in.} \quad \text{and} \quad M = 8P = 80{,}000 \text{ lb.-in.};$$

and the shearing and normal unit-stresses due, respectively, to T and M are

$$s_s = \frac{Tc}{J} = \frac{16T}{\pi d^3} \quad \text{and} \quad s = \frac{Mc}{I} = \frac{32M}{\pi d^3}.$$

(*a*) *Maximum Normal Stress Theory.*—The maximum normal unit-stress resulting from the unit-stresses s_s and s, as shown in Art. 86, is

$$s' = \tfrac{1}{2}s + \tfrac{1}{2}\sqrt{s^2 + 4s_s^2},$$

and according to the maximum stress theory, inelastic action in the material begins when s' exceeds 40,000 lb. per sq. in. Hence the working unit-stress is 20,000 lb. per sq. in. Therefore,

$$20{,}000 = \frac{1}{2}\frac{32M}{\pi d^3} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + \left(\frac{2 \times 16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}(M + \sqrt{M^2 + T^2}),$$

whence

$$d^3 = \frac{16}{\pi 20{,}000}\,(80{,}000 + \sqrt{(80{,}000)^2 + (60{,}000)^2}),$$

and

$$d^3 = \frac{8}{\pi}\,(8 + \sqrt{(8)^2 + (6)^2}) = 45.8 \text{ in.}^3,$$

Therefore

$$d = 3.58 \text{ in.}$$

(*b*) *Maximum Strain Theory.*—According to the maximum strain theory, inelastic action in the material begins when the maximum value of $E\epsilon$ becomes 40,000 lb. per sq. in. And, from Art. 88 the maximum value of $E\epsilon$, assuming m equal to 0.30, is given by the equation

$$E\epsilon = 0.35s + 0.65\sqrt{s^2 + 4s_s^2}.$$

The working value of $E\epsilon$ is $\frac{1}{2} \times 40{,}000$. Thus,

$$20{,}000 = \frac{32}{\pi d^3}\,(0.35M + 0.65\sqrt{M^2 + T^2}).$$

Hence

$$d^3 = \frac{16}{\pi}\,(0.35 \times 8 + 0.65 \times 10) = 47.4 \text{ in.}^3$$

Therefore

$$d = 3.62 \text{ in.}$$

(*c*) *Maximum Shearing Stress Theory.*—According to this theory inelastic action in the shaft begins when the maximum shearing unit-stress s'_s, as given by the equation

$$s'_s = \tfrac{1}{2}\sqrt{s^2 + 4s_s^2},$$

reaches the shearing proportional limit, that is, $0.6 \times 40{,}000$ or 24,000 lb. per sq. in. The working unit-stress then is 12,000 lb. per sq. in. Hence,

$$12{,}000 = \tfrac{1}{2}\sqrt{s^2 + 4s_s^2} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2}.$$

Hence,

$$d^3 = \frac{160}{12\pi}\sqrt{8^2 + 6^2} = \frac{1600}{12\pi} = 42.5 \text{ in.}^3$$

Therefore,

$$d = 3.49 \text{ in.}$$

PROBLEMS

370. In the above problem change the length of the crank from 6 in. to 12 in. and find, by each of the three theories, the diameter of the shaft required.

371. Find, by all three theories of failure of elastic action, the diameter that the steel bolt shown in Fig. 234(*a*) should have in order to resist a load P of 4000 lb. and a load Q of 3000 lb. without developing more than $\frac{1}{2}$ of the maximum elastic strength of the material. The bolt is made of steel having a tensile proportional limit of 30,000 lb. per sq. in. Use a value of 0.25 for Poisson's ratio, and assume the shearing elastic strength to be 0.60 of the tensile elastic strength.

Ans. 0.690 in.; 0.715 in.; 0.715 in.

372. Find, by all three theories of failure of elastic action, the values of the load P and torsional moment T or Qq (Fig. 223) assuming that $P = 2T$ and that the shaft, in resisting the loads, develops 0.65 of the maximum elastic strength of the material. The shaft is made of steel having a tensile elastic limit of 60,000 lb. per sq. in., and the diameter of the shaft is 2 in. Use a value of 0.25 for Poisson's ratio, and assume the shearing elastic strength to be 0.60 of the tensile elastic strength.

373. Prove that, when a cylindrical shaft is subjected to combined bending and torsion, the ratio of the maximum shearing stress s'_s to the shearing yield point will be equal to the ratio of $E\epsilon$ to the tensile yield-point when the ratio of the twisting moment, T, to the bending moment, M, is approximately equal to 1.63, provided that the shearing elastic limit is 0.6 of the tensile elastic limit. (See Problem 362 for the ratio of T to M according to the maximum stress theory.) Use Poisson's ratio $=0.3$.

374. A circular steel shaft having a tensile elastic limit of 36,000 lb. per sq. in. is subjected to a bending moment of 50 ton-in. and a twisting moment of 50 ton-in. Assume that the shearing elastic limit is 0.6 of the tensile elastic limit. Compute by each of the three theories of failure of elastic action the diameter the shaft must have if it develops $\frac{1}{3}$ the maximum elastic strength of the material. Use a value of 0.25 for Poisson's ratio. *Ans.* 4.68 in., 4.64 in., 4.74 in.

REVIEW QUESTIONS AND PROBLEMS

375. Point out and correct the error in each of the following statements:

(*a*) There is only one principal unit-stress at a point in a stressed body and it is always the maximum normal unit-stress at the point.

(*b*) The maximum shearing unit-stress at a point in a stressed body occurs on the plane on which no normal stress acts.

(*c*) Poisson's ratio for a material is the ratio of the longitudinal unit-strain of a bar of the material caused by an axial load, to the lateral unit-strain of the bar.

376. If the three principal unit-stresses at a point in a stressed body are known, how can the maximum shearing unit-stress at the same point be obtained? On what planes (relative to the planes on which the principal stresses exist) does it act?

377. If at a point in a body the three principal unit-stresses are equal in magnitude and of like sign, what is the value of the maximum shearing unit-stress at the point? If one of the three principal stresses has the value of zero and the other two are equal in magnitude and of like sign, what is the maximum shearing unit-stress at the point?

378. At a point in a stressed body a normal unit-stress s exists on a given plane and a shearing unit-stress s_s on the same plane. What unit-stress must act at the same point on a plane at right angles to the given plane?

379. When the state of stress considered in question 378 occurs at a point in a stressed body, what will be the magnitude of the principal stresses at the same point? Draw a sketch showing approximately (in relation to the given plane), on which planes the principal stresses will act. What will be the magnitude of the maximum shearing unit-stress at the same point; show approximately on what planes it will act?

380. In deriving the results asked for in questions 376, 377, and 378, what are the main steps in the procedure? Demonstrate the procedure by proving that if a state

of pure shear exists at a point in a body, normal unit-stresses exist at the same point on planes bisecting the planes on which the shearing unit-stresses occur and that these normal unit-stresses are equal in magnitude to the shearing unit-stresses.

381. The maximum shearing unit-stress at a point in a stressed body is always less than the maximum principal stress at the point. Why is it, then, desirable to know the maximum shearing unit-stress at the point? Explain the maximum shearing stress theory of elastic failure of a material.

382. A steel shaft 2 in. in diameter is subjected to a twisting moment of 2500π lb.-in. and an axial tensile load of 4000π lb. Calculate the maximum tensile and shearing unit-stresses in the shaft. Draw a sketch indicating approximately the planes on which these stresses occur.

Ans. $s'_t = 7380$ lb. per sq. in.; $s'_s = 5380$ lb. per sq. in.

383. The pulley, A (Fig. 236), exerts a constant twisting moment of 14,400 lb.-in. on the shaft B. The total bending load P due to the tensions in the belt on the two sides of each pulley is 800 lb. The diameter of the shaft is 2 in. and the shaft is mounted in flexible bearings. The driven pulley is 18 in. from the right hand support and the distance between supports is 36 in. The shaft is made of cold-rolled steel for which the tensile ultimate strength is 80,000 lb. per sq. in., the tensile yield-point is 40,000 lb. per sq. in., and the shearing yield-point is 24,000 lb. per sq. in. Calculate the factors of safety for the tensile and shearing unit-stresses in the shaft based on the yield-points of the material. *Ans.* $f_t = 2.33$, $f_s = 2.18$.

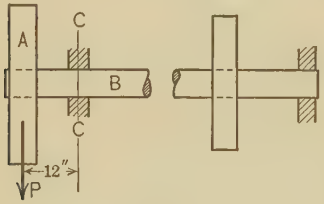

FIG. 236.

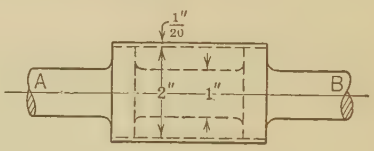

FIG. 237.

384. The steel tube shown in Fig. 237 is welded (or otherwise rigidly attached) to the enlarged portions of the steel shaft. The tube is subjected to an internal fluid pressure that causes a circumferential stress of 20,000 lb. per sq. in. in the tube; the shaft at the same time is subjected to an axial tensile load P of 10,000 lb. at A and B. It is assumed that the shaft and tube between the enlarged portions of the shaft stretch the same amount. Neglecting the effect of the radial stress in the tube and in the shaft on the longitudinal strains of the tube and shaft, calculate the longitudinal unit-stress in the shaft and in the tube. Assume Poisson's ratio to be 0.25.

Ans. 9820 lb. per sq. in. in shaft.; 14,800 lb. per sq. in. in tube.

CHAPTER X

REPEATED LOADS. FATIGUE OF METALS

90. Introduction.—A repeated load is a force that is applied many times to a member, causing stress in the material that continually varies, usually through some definite range. If a stress is developed in a member and is then released the member is said to have been subjected to a *cycle of stress.* Further, if a tensile stress has been developed and when released a compressive stress is developed and this stress is then released, the member is said to have been subjected to a *reversed cycle of stress* or, briefly, to a *reversal of stress*; the reversal of stress is said to be *complete* if the opposite stresses are of equal magnitudes. For example the piston rod of a steam engine that runs at 300 r.p.m. 10 hours per day 300 days per year for 10 years is subjected to 540,000,000 cycles of approximately complete reversals of axial stress; a car axle is subjected to about 50,000,000 complete reversals of bending stress in its normal "life"; a band saw in a normal service of about two months is subjected to about 10,000,000 cycles of stress, the stress in each cycle ranging from approximately zero to a maximum, etc.

Experience and experiments have shown clearly that the safe loads for most structural and machine metal members when the loads on the members are repeated a very large number of times may be much less than the safe static loads for the same members. For example, iron and steel may rupture when subjected to millions of reversals of stress not only when the calculated unit-stress in the material is less than the static ultimate strength of the material but even *when the calculated unit-stress is less than the static proportional limit.*

The failure of a material caused by repeated loads is not accompanied by yielding; the failure is a gradual or progressive fracture. The fracture seems to start at some point in the material at which the stress is much larger than the calculated stress, and this high localized stress seems to cause, or at least is accompanied by, a small crack which gradually spreads, as the stress is repeated, until the whole member ruptures.

For example, in Fig. 238 is shown the section of a ruptured bolt that failed after being subjected to many repetitions of a tensile load. The failure started at the root of the thread, where high localized stress

occurred, and spread inwardly. The dark portions of the cross-section show the area over which the crack spread and the light portion shows the area of rupture over which the material gave way suddenly.

The apparent loss of strength of a material due to repeated stress is frequently called *fatigue*; thus a fatigue failure is a progressive fracture caused by repeated stress. Failures of various machine members due to repeated stress have frequently occurred in service: Railway axles, steam turbine shafts and discs, crank shafts, piston rods, valve rods,

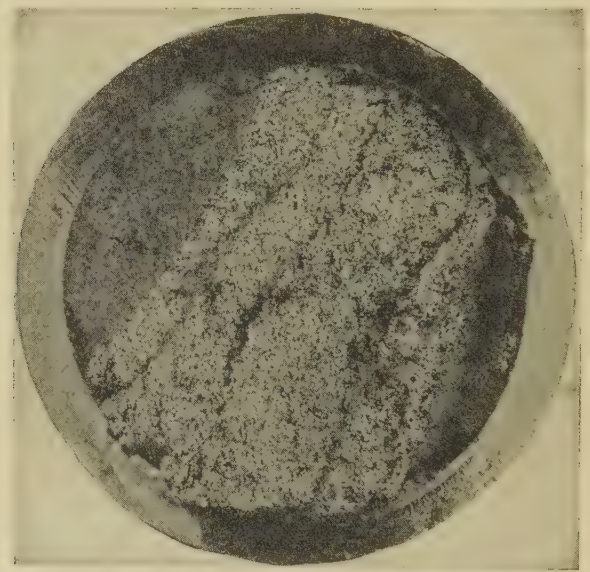

Fig. 238.—View of area of rupture of bolt that failed due to repeated stress, showing evidence of progressive failure. (Obtained by Prof. H. F. Moore.)

springs in automobiles, etc., give much trouble in this respect. The following quotation[1] indicates the importance of so-called fatigue failures.

> "Failures by spreading 'fatigue' cracks constitute a majority of failures in machine industries. It has been stated by Aitchison of England that 95 per cent of broken automobile parts have failed through fatigue. According to Roos, the director of the Swedish Materials Testing Laboratory of Stockholm, at least 80 per cent of all failures of machine parts which have come to his attention have been fatigue failures. The remaining 20 per cent were not static

[1] "Present-Day Knowledge of Fatigue," Proc. Am. Soc. for Test. Materials, Vol. 30, Part I, p. 143.

failures, but involved impact as a cause. Hardly a single failure in machine practice could be attributed to static loading alone. Sir Thomas Stanton has stated 'there can be no doubt that the maximum range of stress of an unlimited number of repetitions is the most valuable strength characteristic of a material which the designer of structures or machines can possess.'"

Experimental Investigations.—In 1870 Wöhler published [2] the results of an extensive series of tests of various grades of iron and steel subjected to repeated direct tensile and compressive loads, to repeated bending loads, and to repeated torsional loads. The tests were carried out for the Prussian Railways during a period of about ten years. They are considered a classic in this field of investigation and until about 1920 were the main source of our knowledge of resistance of material to repeated loads.

The number of cycles of stress to which the material was subjected in Wöhler's tests was usually less than 1,000,000, although a small number of specimens were stressed 10,000,000 times, a very few specimens were stressed 40,000,000 times, and one specimen 132,000,000 times which at 72 cycles per minute took $3\frac{1}{2}$ years. Although Wöhler's experiments gave reliable information for the machines and materials used at that time, the development of high-speed machinery and of alloy and heat-treated steels and non-ferrous alloys have created a need for further experimental investigations. In recent years many important investigations have been made that have added much to our knowledge of the subject.

91. Endurance Limit.—The *endurance limit of a material* is the maximum unit-stress that can be repeated, through a definite cycle or range of stress, an indefinitely large number of times without causing the material to rupture. When the term endurance limit is used without any limiting statement as to range of stress it will be understood to be the endurance limit with completely reversed cycles of stress. The name *fatigue limit* is sometimes used instead of endurance limit. The endurance limit of a material is found as follows.

S–N Diagrams.—If several specimens are cut from the same bar of forged or of rolled steel and most other wrought (rolled or forged) metal alloys and are subjected to repeated complete reversals of stress (see Fig. 240 for description of a repeated-stress testing machine), it will be found that when a specimen is stressed nearly to the ultimate strength of the material in each cycle of stress the specimen will rupture after

[2] A summary of Wöhler's work is given in English in "The Testing of Materials of Construction," by Unwin; published by Longmans Green.

being subjected to a small number of cycles of stress; if a second specimen is tested in the same way but stressed slightly less than the first, a larger number of reversals of stress will be required to cause the specimen to rupture. Now, if a series of such experiments are carried out, the maximum unit-stress in any specimen being somewhat less than in the preceding specimen, the relation between the value of the completely reversed unit-stress, s, and the number of reversals, N, required to rupture the specimen will be found to be represented by a curve similar to that shown in Fig. 239 in which stresses are plotted as ordinates and numbers of reversals as abscissas.

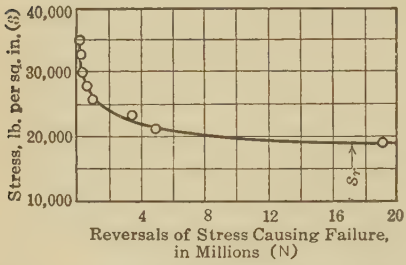

FIG. 239.—An s–N diagram for steel subjected to completely reversed cycles of bending stress.

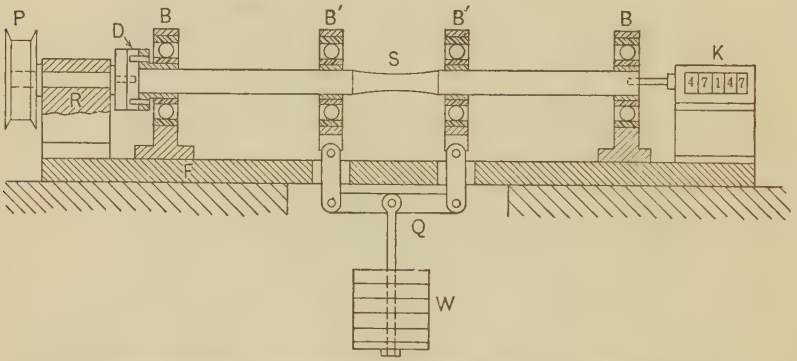

FIG. 240.—Rotating-beam repeated-stress testing machine. This machine produces complete reversals of bending stress by the use of a rotating beam specimen. The specimen S is supported on ball bearings B and is driven by a pulley P. Weights are hung from a second set of ball bearings B', and these weights cause bending stresses in the specimen. The bending stress in the upper fibers of the specimen is compression, and in the lower fibers, tension. As the specimen is rotated the stress for any fiber changes from compression to tension, and the stress is completely reversed. The maximum unit-stress for both tension and compression can be computed from the amount of weights W applied, the dimensions of the specimen, and the distances between bearings; the bending moment is constant for all sections between the two center bearings. A counter K indicates the number of reversals of stress given to the specimen, and when the specimen breaks the counter automatically stops.

The curve in Fig. 239 is called an s–N curve, and the ordinate to the s–N curve where the curve has become approximately horizontal

is taken as a measure of the *endurance limit* of the material as defined above. Thus the endurance limit (s_r) obtained from the curve in Fig. 239 is approximately $\pm$ 19,000 lb. per sq. in. That is, this material will rupture when subjected, in bending, to several million cycles of completely reversed stress if the maximum unit-stress in each cycle is slightly greater than 19,000 lb. per sq. in.

A more convenient way of obtaining an s–N curve and the endurance limit is to plot values of the logarithms of s and N (or the equivalent of this, namely, to plot values of s and N on logarithmic paper). When this is done the s–N curve (Fig. 241) is an approximately straight sloping line until it changes its slope rather abruptly and becomes horizontal. The unit-stress at which the change of slope occurs is taken as the measure of the endurance limit (s_r). Logarithmic plotting makes it possible to show a large range of values on convenient sized paper.

Another convenient method of obtaining an s–N curve and the endurance limit is to plot the unit-stresses as ordinates and the logarithms of the numbers of reversals as abscissae, sometimes called semi-logarithmic plotting. The curves thus obtained are very much the same as those in Fig. 241. Thus, in Fig. 242, are shown s–N diagrams for several grades of steel plotted in two ways: In Fig. 242(a) values of s and N are plotted as Cartesian coordinates and in Fig. 242(b) is shown the semi-logarithmic plotting of the same values.

TABLE 4

NUMBER OF REPETITIONS IN "LIFETIME" OF MEMBERS

Part of Structure or Machine	Approximate number of repetitions of stress in the "lifetime" of the structure or machine.
Railroad bridge, chord members	2,000,000
Elevated railroad structure, floor beams	40,000,000
Railroad rail, locomotive wheel loads	500,000
Railroad rail, car wheel loads	15,000,000
Airplane engine, crankshaft	18,000,000
Car axles	50,000,000
Automobile engine, crankshaft	120,000,000
Line shafting in shops	360,000,000
Steam engine, piston rods, connecting rods and crank-shafts	1,000,000,000
Steam-turbine shafts	15,000,000,000
Steam-turbine blades	250,000,000,000

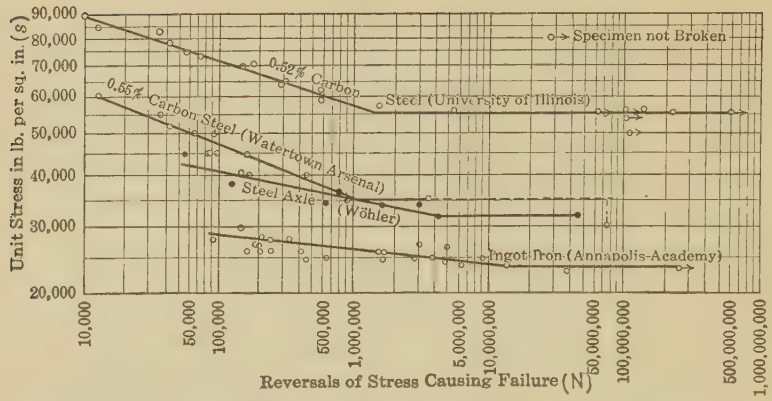

FIG. 241.—Logarithmic s–N diagrams for steel subjected to completely reversed cycles of bending stress.

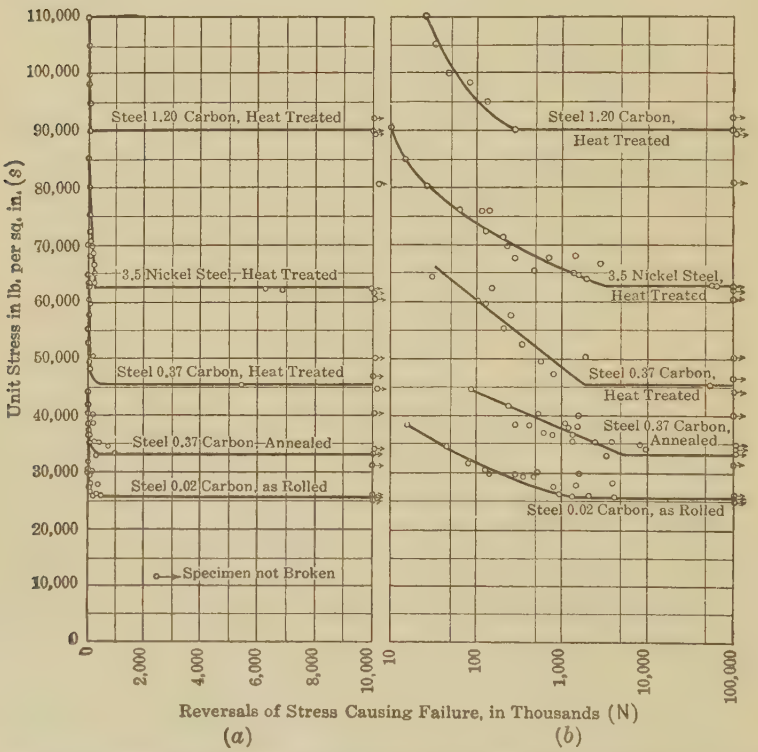

FIG. 242.—s–N diagrams for steel subjected to completely reversed cycles of bending stress; (a) Cartesian plotting; (b) semi-logarithmic plotting.

As shown in Figs. 241 and 242, wrought ferrous metals, if subjected to complete reversals of stress, will usually fail after resisting 1,000,000 to 5,000,000 cycles of stress, when the maximum stress in each cycle is slightly above the endurance limit. (Compare these values with the probable number of repetitions of stress in the " lifetime " of various members as given in Table 4 and determine if those members in which the stress is completely reversed can safely be stressed above the endurance limit.)

The endurance limits of non-ferrous metals are not so sharply defined as are those of wrought ferrous metals and the number of cycles of stress required to determine the endurance limit is usually considerably greater than for ferrous metals.

92. Localized Stress and Fatigue Failure.—When a *ductile* steel specimen is caused to fail by a gradually increasing (static load) there is visible evidence of structural damage (plastic deformation) and of the approach of failure, considerably before the failure occurs. If a specimen of the same material, however, is caused to fail by a repeated load, there is no plastic deformation or other warning of the approach of failure. Thus the *fatigue* failure of a ductile steel is similar to that of a *static* failure of a brittle material; the fractured area of a *ductile* steel specimen that is broken by repeated loads, usually presents a crystalline appearance similar to that of a coarse-grained brittle material, whereas the fractured area of the ductile steel specimen when broken by a static load presents a " silky " or " fibrous " appearance.

These observations suggested the *crystallization theory* of failure of steel due to repeated stress. It was thought that, in service, the repeated application of a load changed the steel from a fibrous ductile material to a crystalline brittle material. It is now known, however, that iron and steel are always crystalline and that the crystallization theory is entirely erroneous.

Localized Stress Theory.—From the engineering viewpoint, probably the most satisfactory explanation of a fatigue failure is the *localized stress theory,* the main features of which may be explained as follows:

In determining the relation between stresses and static loads as expressed by the equations developed in the preceding chapters it was assumed

1. That the material and member were continuous and homogeneous; that is, there existed no discontinuities in the material such as, minute internal flaws, etc., and no sudden or abrupt changes in the external form of the member, and no changes in the properties of the material from point to point throughout the material.

2. And, therefore, that on any section of a stressed member there

was a definite regularity of stress distribution; for example, in a beam the stresses at points on a transverse section varied directly as the distances of the points from the neutral axis. In other words, it was assumed that the intensity of stress at any one point could not attain a value greatly in excess of the stress at neighboring points as would be true of the high localized stress that exists at the edges of an internal crack or flaw, and at the inward projecting corner of an external abrupt change in section such as at the root of a screw thread, etc. It is known, however, that the assumed regularity of stress distribution never exists in structural members; metals for example are composed of crystalline grains whose strength and stiffness vary, and in which there are local concentrations of stress (localized stresses) at various portions of a member that may be much larger than the calculated values based on the assumed regularity of distribution. These localized stresses and irregularities in the stress distribution are due (1) to discontinuities in the material itself such as small flaws, fissures, non-metallic inclusions, etc., at the edges of which high stress exist, and discontinuities in the properties of the materials due to the variation in the strength and stiffness of the crystalline grains from those assumed for the material as a whole. Both of these causes of localized stress will be referred to as *internal discontinuities*, and (2) to *external discontinuities* due mainly to abrupt changes of sections of the body such as occur at fillets, roots of threads, holes, etc., where the stress may be highly localized.

Now when a ductile material is subjected to a *static* load, high localized stresses may be developed without seriously affecting the strength or deformation of the member *as a whole*, and hence the assumption of regularity of stress distribution leads, in general, to reliable results when the load is a static load (or is applied only a few times). But when the load is applied a very large number of times these localized stresses are accompanied by the formation of minute cracks which gradually spread as the stress is repeated until the whole member breaks. The first experimental evidence of the gradual spread of the area of rupture was obtained by Ewing, Humphrey and Rosenhain.[3] They found that when the localized stress in certain crystals becomes sufficiently large the crystals yield by sliding along many planes.

These planes are called *slip planes* and their traces on a polished section of the member are seen under the microscope as dark lines, called *slip lines* or *slip bands* (see Fig. 243). For example, slip lines were detected in some of the crystals of a Swedish iron specimen when subjected to a few complete reversals of stress of $\pm$ 20,000 lb. per sq. in., although the

[3] Philosophic Transactions Royal Society, A, Vol. 200, p. 241, 1903.

(*a*) (*b*)

FIG. 243.—Photo-micrographs showing slip planes in low carbon steel when stressed near to the yield point. (Magnification = 300.) Obtained by Prof. T. Ver at Univ. of Ill. (*a*) Crystal *A* is the only crystal in the group shown in which slip planes are developed. (*b*) After increasing the load, slip planes are also developed in crystal *B*.

TABLE 5

VALUES OF ENDURANCE LIMIT, ULTIMATE STRENGTH, PROPORTIONAL LIMIT, AND OF THE RATIO OF THE ENDURANCE LIMIT TO THE ULTIMATE STRENGTH AND TO THE PROPORTIONAL LIMIT FOR VARIOUS GRADES OF STEEL

(Specimens were subjected to completely reversed cycles of bending stress and were free from abrupt changes of section or surface scratches.)

Description of Steel		Endurance Limit, Lb. per Sq. In., s_r	Ultimate Tensile Strength, Lb. per Sq. In., s_u	Tensile Proportional Limit, Lb. per Sq. In., s_e	Ratio	
Composition of steel	Treatment				$\frac{s_r}{s_u}$	$\frac{s_r}{s_e}$
0.02% C (Ingot Iron)	As received (hot rolled)	25,000	42,400	16,100	0.58	1.56
0.21% C	As received (hot rolled)	34,000	70,000	39,900	.48	0.85
0.20% C	Cold drawn (surface polished after drawing)	41,000	80,800	55,200	.50	.74
0.30% C	Annealed*	30,000	69,900	35,000	.43	.86
0.57% C	Normalized*	33,000	71,900	34,500	.46	.96
0.37% C	Hardened† and Tempered	45,000	94,200	61,500	.48	.73
0.49% C	Hardened† and Tempered	48,000	96,900	67,700	.50	.71
0.77% C	Annealed	39,000	111,100	46,300	.35	.88
0.93% C (Spg. Steel)	Annealed‡	30,500	84,100	28,000	.36	1.09
0.93% C " "	Hardened and Tempered	56,000	97,100	84,800	.58	0.67
0.93% C " "	Hardened	98,000	188,300	106,500	.52	.92
1.20% C	Normalized‡	50,000	116,900	55,300	.43	.90
1.20% C	Hardened and Tempered	92,000	179,900	100,700	.51	.91
3.5% Nickel	Hardened and Tempered	63,000	118,000	86,400	.53	.73
3.5% Nickel	Annealed	54,000	101,600	60,800	.53	.89
Chrome-Nickel 3.3% Ni., 0.24% C, 0.87% Cr	Hardened and Tempered	68,000	138,700	122,600	.49	.57
Chrome-Nickel 3.3% Ni., 0.24% C, 0.87% Cr	Annealed	49,000	87,300	56,700	.56	.86
Nickel-Molybdenum	Hardened and Tempered	57,000	133,300	100,000	.43	.57
Chrome-Molybdenum	Hardened and Tempered	67,000	141,400	112,500	.48	.60
Chrome-Vanadium	Hardened and Tempered	67,000	146,600	114,000	.46	.59
Selico-Manganese (Spring steel)	Softer than is usual for springs	62,000	157,400	100,000	.40	.62

* Annealing consists in heating the steel above the upper critical temperature (about 800° C.) and then cooling the steel very slowly, as for example, cooling in the furnace after the fire is drawn. Normalizing differs from annealing in that the rate of cooling is somewhat faster, the material is usually drawn from the furnace and allowed to cool in the air.

† Hardening consists in heating the steel above its upper critical temperature and cooling it rapidly by quenching it in water or oil. Tempering consists in heating the material, after it has been quenched, to 450° C., more or less, to reduce the hardness slightly and to give the material a more uniform internal structure.

‡ This steel would seldom be used except in the hardened and tempered condition; it is included in the table mainly to show the effect of heat treatment.

static yield-point and ultimate strength of the material were 31,600 and 52,800 lb. per sq. in., respectively; after more reversals of stress were applied additional slip lines appeared and the original ones broadened. Finally, various groups of slip lines united forming a visible crack which gradually extended until rupture occurred.

93. Values of Endurance Limits with Completely Reversed Stress.—Values of the endurance limits of several grades of steel when subjected to completely reversed *bending* stress are given in Table 5. Values of the static tensile proportional limits and of the tensile ultimate strengths of the steel are also given in the same table.

The values in Table 5 show that the endurance limit for complete reversals of bending stress may be considerably below the static proportional limit of the material. In other words, the *ultimate* resistance of steel to repeated loads is often considerably less than the *static elastic* resistance of the material. Further, if a heat treatment greatly increases the static elastic strength it does not necessarily follow that the endurance limit is raised proportionally. In fact, tests show clearly that the endurance limit of steel bears a much more constant relation to the ultimate strength than to the elastic limit or proportional limit, and that for most rolled or forged steels the *endurance limit with completely reversed bending stress is approximately* 0.5 *of the static tensile ultimate strength* ($s_r = 0.5 s_u$) as may be seen by referring to Table 5. This relation, however, does not hold for the non-ferrous metals, and is less reliable for the harder steels than for the softer steels. For endurance limits of light alloys (aluminum and magnesium alloys), see Proc. American Society for Testing Materials, Vol. 33, Part I, page 364.

Endurance Limit for Complete Reversals of Direct Stress.—The endurance limit of a material when subjected to complete reversals of axial direct stress (stress varying from a direct tensile to an equal direct compressive stress) is found to be approximately the same as the endurance limit of the same material when subjected to complete reversals of bending stress, provided that great care is taken to obtain an axial load.

However, in most tests in which a reasonable effort is made to obtain an axial load the endurance limit with completely reversed direct stress is about two-thirds of that for reversed bending stress. In other words, the bending action due to the eccentricity of loading apparently caused a maximum stress $s = \frac{3}{2}\frac{P}{a}$ instead of $s = \frac{P}{a}$ which would have been developed had the load been truly axial.

Therefore, in problems involving complete reversals, of supposedly axial loads applied to a member *without* abrupt change of section it will

be well to calculate the significant stress (particularly when it occurs at an outer fiber) by use of the formula

$$s = 1.5\frac{P}{a},$$

basing a working value for s on the endurance limit as obtained from bending tests.

Endurance Limit for Complete Reversals of Torsional Shearing Stress.—The resistance of material to repeated *shearing* stress is of importance in some machine members such as torsional springs, shafts, etc., and although the number of repeated stress tests with reversed torsion is much less than with reversed bending, the tests are reasonably consistent in showing that the endurance limit of steel with complete reversals of torsional shearing stress, denoted by s_{rs}, is 0.50 times the endurance limit with complete reversals of bending stress. Thus,

$$s_{rs} = 0.50 s_r.$$

This relation is also applicable to a number of the non-ferrous metals.

94. Working Stress Based on Endurance Limit.—The endurance limit of steel subjected to millions of cycles of stress is really the fatigue *ultimate* strength of the material since if the unit-stress developed in the material is greater than the endurance limit the material will rupture. Further, there is no unit-stress, below the endurance limit, at which the beginning of inelastic action or structural damage can be detected, whereas with static loading the beginning of structural damage can be detected at stresses much below the static ultimate strength of the material since the yield-point indicates the beginning of structural damage caused by static loads. It seems clear, therefore, that the working stress for members subjected to repeated stress should be based on the endurance limit of the material.

If, however, the endurance limit of the steel is not known, an approximate value of the endurance limit may be found from the static ultimate strength of the steel since tests have shown, as stated in Art. 93, that the endurance limit of *steel* for completely reversed cycles of stress may be taken with reasonable accuracy as 0.50 of the static tensile ultimate strength of the steel.

Further, if the stress is not completely reversed, an approximate value of the endurance limit for a given range of stress may be obtained from the endurance limit with completely reversed stress by means of equation (198).

It is important to note that the values of the endurance limits found from the usual laboratory tests are obtained with a specimen having a very gradual change in section and having a polished surface finish, and hence the endurance limit of the *specimen* may be considered to be the endurance limit of the *material*, the internal discontinuities being assumed constant for a given material. Again, it is only when these conditions exist that the endurance limit may be assumed to be 0.50 of the static tensile ultimate strength of the material.

The value of the so-called factor of safety to be used with repeated stress depends, of course, on the conditions of service; but in general a somewhat larger value should be used than for static loading on a ductile material. For example, if a factor of safety of 2 based on the yield point is used for structural steel when subjected to static loads, a value of 3 based on the endurance limit might well be selected for the same steel if the member is to be subjected to many millions of repetitions of stress.

95. Stress Concentration Factor. Method of Calculating Localized Stress.—The results of several independent experimental investigations have shown that high localized stresses occur at abrupt changes of section of a member when the member is resisting a load. Consequently, a repeated stress specimen having an abrupt change of section such as a square corner, keyway, screw thread, groove, etc., is found to fail when the *calculated* stress (found by use of the ordinary equations of mechanics of materials) is lower than the endurance limit of the material as found from specimens of the *same material* that are free from abrupt changes of sections; the actual unit-stresses causing failure in the two specimens are equal. The ratio, then, of the endurance limit of the *material* to the endurance limit of the specimen with the abrupt change of section is a measure of the localized stress due to the abrupt change of section. This ratio is called the *stress concentration factor* for the given external discontinuity. For example, if the endurance limit of specimens free from abrupt changes of section is found to be 30,000 lb. per sq. in. and the endurance limit of specimens having a given abrupt change of section is found to be 15,000 lb. per sq. in., then the actual (localized) stress in the latter specimens is considered to be twice the calculated stress; that is, the stress concentration factor is 2. Thus, repeated stress tests may be used to determine approximate values of localized stresses.

Values of stress concentration factors for several kinds of external discontinuities are given in Table 6. The value of a stress concentration factor, however, depends on the kind and condition of the material and not alone on the type of abrupt change in section. For example,

a fillet in a member made of a fine-grained high-strength heat-treated steel usually has a greater influence on the fatigue strength of the member than does the same sized fillet on a similar member made of ordinary low-strength relatively coarse-grained steel. The values in Table 6 therefore must be considered to be approximate values.

TABLE 6

STRESS CONCENTRATION FACTORS

Method of Stressing and Type of Discontinuity	Stress Concentration Factor (k)
Bending or Direct Tension:	
Concentric Groove around Small Cylindrical Shaft:	
Ratio of radius of groove to diameter of shaft—	
0.1	2.0
0.5	1.6
1.0	1.2
2.0	1.1
Fillet:	
Ratio of radius of full fillet to depth of member—	
$\frac{1}{16}$	1.75
$\frac{1}{8}$	1.50
$\frac{1}{4}$	1.20
$\frac{1}{2}$	1.10
Square corner	2.0
Sharp V-thread	3.0
Whitworth thread	2.0
U. S. standard thread	2.5
Hole:	
Ratio of diameter of hole to depth of member—	
$\frac{1}{10}$ to $\frac{1}{3}$	2.0
Surface finish produced by lathe tool (scratches due to grit in bearings)	1.2
Torsion:	
Fillet in Cylindrical Shaft:	
Ratio of radius of fillet to radius of smaller shaft—	
0.02	1.8
0.10	1.2
0.20	1.1

Method of Calculating Localized Stress.—In calculating the significant stress in a member having some form of abrupt change in section, when the load is repeated many times, the formulas developed in the preced-

ing chapters are modified by introducing a stress concentration factor. Thus for axial, torsional, and bending loads we have, respectively,

$$s = k\frac{P}{a}, \quad s_s = k\frac{Tc}{J}, \quad \text{and} \quad s = k\frac{Mc}{I},$$

where k is a stress concentration factor. The stress distribution at an abrupt change in section in a tension and a flexural member may be pictured approximately as in Fig. 244.

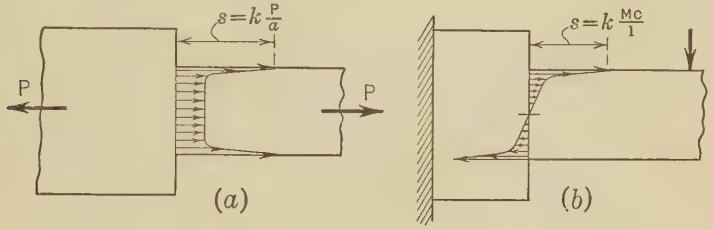

FIG. 244.—Stress concentration at abrupt change in section.

The above formulas should also be used for very brittle material subjected to static loads. But for most of the *ductile* metals the localized stress at abrupt changes of section may be disregarded (k may be taken as unity) when the material is subjected to static loads; for, the material at the point where a localized stress occurs will yield, and the stress at this point will be relieved, thereby transmitting some of the excess stress to the surrounding, less-stressed material; and the resistance of the member as a whole will not be seriously affected by this action.

It is assumed throughout this chapter that the repeated loads are not applied with impact. If a member is subjected to repeated impact loads some additional allowance must be made for the effect of the impact, depending on the conditions of the problem.

Again, some machine members in service are subjected to repeated stresses greater than the endurance limit and are not expected to resist these stresses indefinitely, as for example, band saws for wood, wire rope running on sheaves, ball bearings, gun barrels, etc.

ILLUSTRATIVE PROBLEM

385. Find the maximum safe axial load P that can be applied to the steel bar shown in Fig. 245, assuming the load to be (*a*) a gradually applied (static) load, and (*b*) a completely reversed repeated load. The yield point of the material is 50,000 lb. per sq. in., the tensile ultimate strength is 85,000 lb. per sq. in., and the endurance limit for completely reversed bending stress is 42,000 lb. per sq. in. Use a factor of safety of 2.5 based on the yield point for static loading, and a factor of 3.5 based on the endurance limit.

Solution.—(*a*) *Static Loading.* The significant stress with static loading will occur in the smaller part of the member, and the concentration of stress at A may be neglected. Thus

$$s = \frac{P}{a} \quad \text{or} \quad P = \frac{50{,}000}{2.5} \times 4 = 20{,}000 \times 4 = 80{,}000 \text{ lb.}$$

(*b*) *Repeated Loading.* The significant stress for repeated loading is either at A or at B, and is a localized stress.

If at A:

$$s = k\frac{P}{a} \text{ in which } k = 1.75 \text{ (see Table 6).}$$

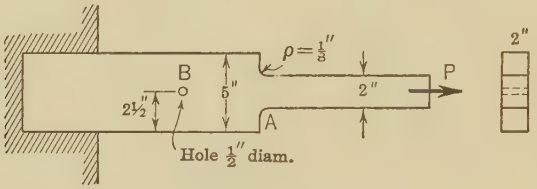

FIG. 245.

But as pointed out in Art. 93 a *nominal* axial load may produce a stress $s = 1.5\dfrac{P}{a}$ in a member free from an abrupt change in section. Therefore unless the load is known to be nearly a true axial load, it would be wise to consider the significant stress to be

$$s = 1.75\left(1.50\frac{P}{a}\right) = 2.62\frac{P}{a},$$

$$P = \frac{42{,}000 \times 4}{3.5 \times 2.62} = \frac{12{,}000 \times 4}{2.62} = 18{,}300 \text{ lb.}$$

If at B:

$$s = k\frac{P}{a_1}, \text{ where } k = 2.0 \text{ (see Table 6).}$$

Thus,

$$P = \frac{12{,}000 \times 5 \times 2}{2} = 60{,}000 \text{ lb.}$$

Therefore, the repeated stress at point A is the most significant stress and the maximum safe load is a repeatedly reversed load of 18,300 lb.

PROBLEMS

386. The crank pin (Fig. 246) has a diameter, d, of 5 in. and a length of 6.25 in., these dimensions being determined from the allowable bearing pressure. The shaft is made of the normalized 0.57 per cent carbon steel listed in Table 5. The radius of the fillet at section AA is $\frac{1}{32}$ in. The connecting rod exerts a total pressure of 25,000 lb. on the pin. The speed of the shaft is 120 r.p.m., and the engine runs ten hours per day and three hundred days per year. If the pressure on the pin is assumed to be uniformly distributed and the stress concentration factor for the fillet is 2, will

the pin probably fail, due to repeated stress, in a service of ten years? If not, what is the factor of safety based on the endurance limit? *Ans.* No; 2.6.

387. The rods which operate the slide valves of engines frequently break. One method of construction is shown in Fig. 247. Estimate a safe working stress for the rod if the material is hot-rolled low-carbon steel (about 0.20 per cent carbon, Table 5). If the threads on the rod are U. S. standard threads, find the working value of the load P, assuming the cross-sectional area of the bar to be 2 sq. in.

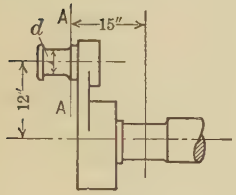

FIG. 246.

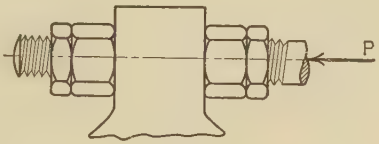

FIG. 247.

FIG. 248.

388. Fatigue failures frequently occur at the corners of cotter holes. If a bar having a rectangular cotter hole (Fig. 248) with square corners is subjected to complete reversals of direct stress, and the bar is made of 0.40 per cent carbon steel having a tensile ultimate strength of 80,000 lb. per sq. in., what working unit-stress should be used in the design of the bar? What working axial load, P, should be applied to the bar if the cross-sectional area of the bar is 4 sq. in.?

389. A circular shaft having a constant diameter of 2 in. is to be subjected to several million complete reversals of bending stress. The shaft is turned with a lathe tool. What is the maximum allowable bending moment that should be applied to the shaft, if the shaft be made of about 0.50 per cent carbon steel, hardened and tempered (see Table 5)? *Ans.* 10,500 lb.-in.

390. A rectangular bar having a cross-section 2 in. deep by 6 in. wide is used as a simple beam as indicated in Fig. 249. The beam contains a hole $\frac{3}{4}$ in. in diameter as shown in the figure. The bar is made of steel having a yield-point of 50,000 lb. per sq. in., a tensile ultimate strength of 80,000 lb. per sq. in. and an endurance limit for completely reversed bending stress of 40,000 lb. per sq. in. Find (*a*) the maximum safe static load P assuming a factor of safety of 2 based on the yield-point, and (*b*) the maximum safe completely reversed repeated load assuming a factor of safety of 3 based on the endurance limit.

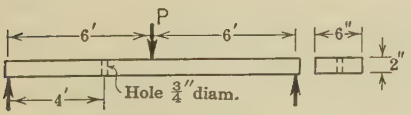

FIG. 249.

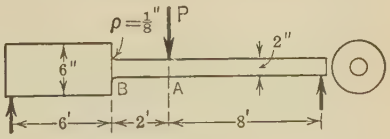

FIG. 250.

391. The steel in the beam shown in Fig. 250 has a yield-point of 40,000 lb. per sq. in., a tensile ultimate strength of 75,000 lb. per sq. in. and a percentage of elongation of 30 in 8 in. The endurance limit of steel for completely reversed bending

stress is approximately one-half ($\frac{1}{2}$) of the tensile ultimate strength. Estimate the maximum safe load P that can be applied to the beam assuming that the load (a) is a static load, (b) a completely reversed repeated load. Use a factor of safety of 2 based on the yield-point for static loading, and a factor of safety of 3 based on the endurance limit for repeated loading. *Ans.* (a) 327 lb.; (b) 156 lb.

392. What should be the diameter d of the steel shaft shown in Fig. 251 if the shaft is subjected to a completely reversed repeated twisting moment of 8000 lb.-in. The endurance limit of the material for completely reversed bending stress is 40,000 lb. per sq. in. and the endurance limit for completely reversed torsional shearing stress is approximately one-half ($\frac{1}{2}$) that for bending stress. Assume a factor of safety of 4 which is considered to be sufficient to allow for some impact.

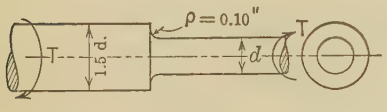

FIG. 251.

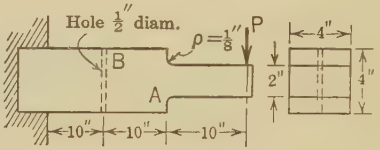

FIG. 252.

393. Estimate the maximum safe load that can be applied to the cantilever beam shown in Fig. 252 assuming the load to be a repeated completely reversed load and assuming a factor of safety of 3 based on the endurance limit. The tensile ultimate strength of the steel of which the beam is made is 90,000 lb. per sq. in. *Ans.* P = 2290 lb.

394. The cantilever beam shown in Fig. 252 is made of a very brittle material (porcelain or plaster of Paris) having a tensile ultimate strength of 400 lb. per sq. in. Estimate the maximum safe load P, assuming the load to be gradually applied and the factor of safety to be 6 based on the ultimate strength.

96. Effect of Range of Stress.—The investigations of Wöhler and the supplementary work of Bauschinger showed that the number of cycles of stress required to cause rupture depends on the range of stress. For example, if the unit-stress in each cycle varies from a tensile stress of 30,000 lb. per sq. in. to a compressive stress of 30,000 lb. per sq. in. (a range of stress of 60,000 lb. per sq. in.), fewer repetitions are necessary to cause rupture than if the unit-stress in each cycle varies from zero to 30,000 lb. per sq. in. although the maximum unit-stress is the same for each of these ranges of stress. A formula based mainly on the results of Wöhler's experiments, that expresses the relation between the endurance limit of a material and the range of stress was developed independently by Goodman[4] and by J. B. Johnson,[5] and will here be referred to as the Goodman-Johnson formula. It may be derived as follows:

Goodman-Johnson Formula.—In Fig. 253 the minimum unit-stresses

[4] Mechanics Applied to Engineering, p. 634.

[5] Johnson's Materials of Construction, 7th Edition, p. 781.

($s_{\text{min.}}$) or lower limits of the various ranges of stress (Δs) which, in Wöhler's experiments, resulted in failure after the application of about 4,000,000 cycles of stress were plotted as ordinates, the abscissas being selected arbitrarily so that a straight line (EOC) would connect the ends of the ordinates. It was then found that if the corresponding maximum unit-stresses ($s_{\text{max.}}$) or upper limits of the ranges of stress were plotted as ordinates the ends of the ordinates fell approximately on the straight line CAD (Fig. 253), where OA is equal to one-half of the static ultimate strength of the material ($\frac{1}{2}s_u$), and MD and ME are equal to one-third

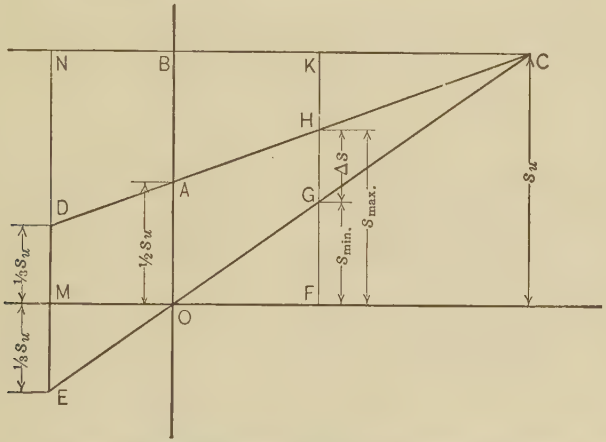

FIG. 253.—Goodman-Johnson diagram. Effect of range of stress on endurance limit of steel.

of the static ultimate strength ($\frac{1}{3}s_u$). Thus, for complete reversals of stress the endurance limit was found by Wöhler to be equal to about $\frac{1}{3}s_u$,[6] and for a range of stress with the minimum stress equal to zero the endurance limit was equal to about $\frac{1}{2}s_u$.

Now from Fig. 253 we have,

$$ND = DE, \quad BA = OA, \quad KH = HG = \Delta s, \text{ etc.}$$

Therefore,

$$\Delta s = FK - FH = s_u - s_{\text{max.}}. \quad . \quad . \quad . \quad . \quad . \quad (197)$$

But $s_{\text{max.}}$ is the endurance limit corresponding to the range of stress Δs. If then s'_r is used to denote the endurance limit, equation (197) may be written

$$s'_r = s_u - \Delta s.$$

[6] As already noted, the more recent tests show that this value is nearly $\frac{1}{2}s_u$, the higher ratio being due probably, in part at least, to more generous fillets and better surface finish.

It is frequently more useful to have the endurance limit expressed in terms of the ratio of the minimum unit-stress to the maximum unit-stress $\left(\frac{s_{\text{min.}}}{s_{\text{max.}}}\right)$ than in terms of the range of stress. Thus, equation (197) may be transformed as follows:

$$s'_r = s_u - \Delta s = s_u - (s_{\text{max.}} - s_{\text{min.}})$$

$$= s_u - (s_{\text{max.}} - q s_{\text{max.}}), \quad \text{where } \frac{s_{\text{min.}}}{s_{\text{max.}}} = q.$$

Therefore,

$$s'_r = \frac{s_u}{2 - q} = \frac{\frac{1}{2}s_u}{1 - \frac{1}{2}q}, \quad \ldots\ldots\ldots\ldots \quad (198)$$

which will be called the Goodman-Johnson formula.

The number of experiments made by Wöhler to determine the effect of range of stress was not large; recent experiments, as already noted, indicate that the Goodman-Johnson formula gives results on the side of safety.

Soderberg [7] found that test data may be interpreted approximately as follows: If a ductile metal is subjected only to a steady normal stress, the tensile (or compressive) yield-point is the maximum useful strength of the material, represented by OA or s_y in Fig. 254. Further, if the same material is subjected only to completely reversed normal stress the endurance limit in reversed bending or direct stress is the maximum useful strength of the material, represented by OB or s_r in Fig. 254. If now a steady stress, s_m (less than s_y) is present in the member, the maximum completely reversed stress that can be superimposed on s_m without causing a fatigue failure is represented by the ordinate CD to the straight line from A to B. That is, the range of stress would be from $s_{\text{min.}} = s_m - CD$ to $s_{\text{max.}} = s_m + CD$ in which $s_{\text{max.}}$ is the endurance limit, denoted by s'_r for the given range of stress. For example, if the initial steady stress s_m is $\frac{2}{3}s_y$ then $CD = \frac{1}{3}s_r$, and hence the endurance limit for the range of stress from $\frac{2}{3}s_y - \frac{1}{3}s_r$ to $\frac{2}{3}s_y + \frac{1}{3}s_r$ is $s'_r = \frac{2}{3}s_y + \frac{1}{3}s_r$.

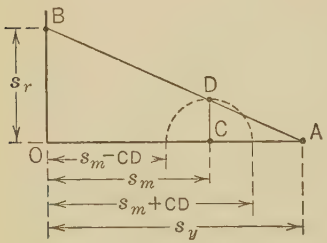

FIG. 254.

[7] Trans. A.S.M.E., Applied Mechanics Division, July–Sept., 1933, Vol. 1, No. 3. *APM*–55–16.

PROBLEMS

395. A hardened and tempered chrome-nickel crank shaft of a single-acting gas engine is subjected to repeated applications of a torsional shearing stress which varies approximately from zero to a maximum ($q = 0$); bending stresses may be neglected. If the shaft is 2 in. in diameter, what maximum torque should be allowed? Use values in Table 5.

396. A special chrome-vanadium heat-treated steel having an ultimate tensile strength of 200,000 lb. per sq. in. and proportional limit of 150,000 lb. per sq. in. is used in a helical spring on an automobile truck. The stress in the spring is mainly shearing stress (see Art. 66) and varies from a small value due to the weight of the truck to a maximum, ($q = 0$). The maximum stress is not constant, however, since the impact loads vary greatly. It may be assumed that the spring is subjected frequently to overload stresses. Estimate the endurance limit and find a working stress, assuming that the working stress should not be greater than one-fourth of the endurance limit. *Ans.* $s'_{rs} = 75{,}000$ lb. per sq. in.; $s_w = 18{,}750$ lb. per sq. in.

397. A simple beam has a span of 20 in. and is made of hardened and tempered chrome-nickel steel (see Table 5). It is loaded at the center of the span with a concentrated load which varies repeatedly from 1000 to 3000 lb. Using a factor of safety of four based on the endurance limit, estimate a safe working stress for the material as used in this member making use of the method illustrated in Fig. 254.

Ans. 24,200 lb./in.^{2}

97. Corrosion-Fatigue.—The endurance limit of metals when the metal is stressed in the presence of a corroding agent such as water may be greatly reduced. The following quotation from the Report of the Research Committee on Fatigue of Metals of the American Society for Testing Materials, Vol. 32 (1932) Part 1, p. 139, gives the pertinent facts concerning corrosion-fatigue.

> " In 1913, Parker Haigh at the Royal Naval Academy at Greenwich noted a remarkable series of failures in railway passenger carriage axles at a section quite removed from those in which fatigue failures usually occur. At this section the axle was exposed to blown spray from the lavatories, and when the lavatory vents were arranged to throw the spray clear of the axles, the breakages stopped. Later during the World War, he further observed similar failures in wire ropes which were exposed to vibration while under sea water. In 1926, McAdam reported an extensive series of tests of rotating-beam fatigue specimens exposed to a stream of water; both fresh water and salt water were tried. He found a very marked reduction of endurance limit under the influence of the stream of water. There seemed to be no very marked general corrosion of the steel, but examination under the microscope showed corrosion pitting over small areas. The following quotation from McAdam gives his conclusions to the nature of this phenomenon:

'The conclusion was reached that corrosion-fatigue is due to the accelerating influence of stress on corrosion. Pits formed under simultaneous stress and corrosion are sharper and deeper than pits formed in the same time under stressless corrosion.

'The corrosion-fatigue process begins with corrosion pitting under the influence of the initial stress. As pitting proceeds, the actual stress rises, because of increasing stress concentration at the bottom of corrosion pits. Eventually the actual stress exceeds the endurance limit and the metal is subjected to ordinary fatigue. In the first part of this second period of corrosion-fatigue, however, the progress of pitting is due entirely to corrosion under the influence of the increasing actual stress. Late in the second period, fatigue cracks start at the bottom of corrosion pits and proceed until the metal fails.

'Even at high cycle frequency, therefore, fatigue does not enter into the process until late in the second period of corrosion-fatigue. At low cycle frequency, pitting may proceed entirely by corrosion, until the weakened section fails by tensile fracture.'

The actual visible corrosion of a specimen which fails by corrosion-fatigue is very slight; the damage is due to *sharp pitting* rather than to general corrosion and consequent reduction of section."

Experimental data are not available for determining the extent to which the endurance limit is reduced for many materials and corroding media, but some tests indicate that the endurance limit of steel may be reduced more than 50 per cent when the repeated stress is developed in the presence of salt water.

REVIEW QUESTIONS AND PROBLEMS

398. Correct the following statement: If a steel member fails to perform its function properly when the member is subjected to repeated reversals of stress, the failure results from yielding (excessive permanent distortion) of the member.

399. Fill in the missing words in the following statement. Localized stresses that may be neglected without introducing serious error in stress calculations in ductile material subjected to static loads cannot be considered to be negligible when the same material is subjected to a large number of reversals of stress, because a ———— is likely to start at the point of high localized stress and to gradually ———— as the stress is repeated until the area of cross-section of the member is greatly reduced and the member ————.

400. Two of the following statements are false; correct the statements. (*a*) The endurance limit of a material for completely reversed cycles of stress is the maximum unit-stress that can be reversed an indefinitely large number of times without causing the material to break (fracture). (*b*) In testing to obtain data for determining the

endurance limit of a material only one specimen is needed. (*c*) In an s–N diagram for completely reversed cycles of stress the ordinates represent the range of stress in each cycle and the abscissas represent the numbers of cycles of stress that caused the specimen to break. (*d*) The test specimens used for determining the endurance limit must be free from abrupt changes in section in order that the maximum stress to which the specimen is subjected can be accurately calculated.

401. The endurance limit for a certain grade of steel as found from specimens subjected to completely reversed bending stress is 36,000 lb. per sq. in. The yield point of the steel as found by the usual static test is 45,000 lb. per sq. in. An axle is made of this steel. The axle has a square shoulder (abrupt change of section) at the section of maximum moment, for which the stress-concentration factor is 2.0, and the section modulus of the shaft at the dangerous section is 12.0 in.^{3} If in service the axle will be subjected to many millions of reversals of bending stress, calculate a working or safe value of the maximum bending moment to which the shaft should be subjected, using a so-called factor of safety of 3 based on the endurance limit.

Ans. 72,000 lb.-in.

402. A steel rod is subjected to a supposedly axial load P which causes completely reversed cycles of stress. V-threads are cut round the rod, the diameter at the root of the threads being $\frac{1}{2}$ in. If the rod is subjected to many millions of cycles of stress, determine a safe or working value of P. The value of the endurance limit of the material as found from tests of specimens subjected to completely reversed *bending* stress is 32,000 lb. per sq. in. Consult Table 6. *Ans.* $P = 465$ lb.

CHAPTER XI

IMPACT AND ENERGY LOADS

98. Introduction.—In the preceding chapters, the loads acting on the members were assumed to be gradually applied (without impact). To state the same idea in other words, the bodies that applied the loads were not in motion when they came in contact with the resisting member, and hence they delivered no kinetic energy to the resisting member.

Members of engineering structures and machines, however, frequently must resist loads that are applied by moving bodies, and the kinetic energy of these moving bodies must be absorbed by the resisting member, thereby developing stresses and deformations in the member.

There are two methods of determining the stresses and deformations developed in the resisting member: (1) To estimate the maximum pressure or force exerted by the moving body on the resisting member; this force is then considered to be a static load and is used in the equations developed in the preceding chapters. (2) To estimate the energy that is absorbed by the resisting member and from this value of the energy determine the stresses and deformations by use of the equations developed in this chapter.

In the first method the load is a force and is, of course, expressed in pounds, tons, etc.; it is called an impact load, and the method is frequently called the *equivalent-static-load* method. Whereas, in the second method the load is considered to be a quantity of energy and is therefore called an *energy load*; it is expressed in foot-pounds, inch-tons, etc.

The determination of the stresses in structural and machine members that are subjected to impact or energy loads is less definite and satisfactory than in members subjected to static loads, for the reason, that, in general, the uncertainty as to the value of an impact load or of an energy load is greater than that of a static load. The design of members subjected to energy loads, therefore, is based more directly on the study of existing machines and structures than is the design of members subjected to static loads.

When the energy delivered to the resisting member is relatively small the equivalent-static-load method is the one more commonly used in determining the stresses in the member. Thus the load caused by a moving train on a bridge, by a moving crowd of people on a stadium, etc., is assumed to be equivalent to a static load composed of the weight of the moving body (called the live load) plus a static load assumed to be equivalent (in producing stress) to the dynamic or impact effect of the live load. The total equivalent static load, then, is the weight of the structure (called the dead load) plus the live load plus the product of an impact factor and the live load.

If, on the other hand, the energy delivered to the resisting member is relatively large, the energy-load method may be the more useful one, for, it will be found that *the dimensions of the resisting member and the properties of the material in the member that give it maximum resistance to an energy load are quite different from those that give the member maximum resistance to a static load.*

99. Calculation of Energy Delivered to Resisting Member.—As a rule, only a part of the energy of the moving body that delivers the energy to the resisting member is absorbed or stored in the resisting member; some of the energy is spent in each of the following ways:

(*a*) In causing stresses and deformations throughout the moving body itself.

(*b*) In causing local stresses and deformation of both bodies at the surface of contact, especially if the velocity of the moving body is large when it comes in contact with the resisting body, and if the resisting body is relatively large.

(*c*) In deforming and in moving the external supports to which the resisting member is attached.

Thus, for example, an airplane when landing possesses kinetic energy which is delivered mainly to the wheels and axle on which the wheels are mounted, and these parts are constructed so that they can absorb a large amount of the energy. But, a considerable part of the energy of the airplane is absorbed as stated under (*a*) and (*c*) above, that is, in causing deformation of the frame of the airplane and in deforming the tires and spokes of the wheels and in causing depressions in the ground.

Kinetic Energy of Bodies.—If the moving body that delivers the energy has a motion of translation or of rotation, the kinetic energy of the body may be found from the expressions given below; from this kinetic energy must be subtracted the amount of energy assumed to be lost, in order to obtain the energy load absorbed by the resisting member.

Translation.—The kinetic energy E_k, of a body having a motion of translation is

$$E_k = \frac{1}{2} Mv^2 = \frac{1}{2} \frac{W}{g} v^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

in which v is the velocity of the body and M is the number of units of mass in the body found by dividing the weight, W, of the body by g, the acceleration which the weight causes when it is the only force acting on the body $\left(M = \frac{W}{g}\right)$. When W is expressed in pounds, g in feet per second per second, and v in feet per second, E_k will be expressed in foot-pounds. Further, if the body acquires the velocity v by falling freely from rest through a distance of h feet, then v^2 is equal to $2gh$ and hence $E_k = Wh$. (The value of g is 32.2 ft. per sec.^{2}, approximately.)

Rotation.—If the body that delivers the energy load has a motion of rotation about a fixed axis, its kinetic energy E_k may be found from the expression

$$E_k = \tfrac{1}{2} I_0 \omega^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

in which ω is the angular velocity of the body and I_0 is the moment of inertia of the body about the axis of rotation; I_0 is expressed in terms of the mass $\left(\frac{W}{g}\right)$ of the body and the dimensions of the body. Thus, for a solid cylinder that rotates about its geometric axis $I_0 = \frac{1}{2} \frac{W}{g} r^2$, where r is the radius of the cylinder. If W is expressed in pounds, g in feet per second per second, the dimensions of the body in feet, and ω in radians per second, E_k will be expressed in foot-pounds. The values of I_0 for bodies of other shapes may be found in books on analytical mechanics and in engineers' handbooks.

100. Stress in Bar Due to Axial Energy Load.—Let an energy load U be applied to a bar so that the bar is subjected to an axial tensile stress. Such a load occurs in machines of various types as will be discussed later; the method of estimating the value of U is given in the preceding article; namely, the value of U is the kinetic energy of the moving body less the part of this energy that is lost in ways stated in Art. 99.

The problem is to find the maximum unit-stress in the bar caused by the energy load U. The assumption [1] is made that a material

[1] This assumption probably is not true if the load is applied with extreme suddenness and acts on the body only a very short period of time, causing a " momentary "

when resisting an energy load acts in the same way as when resisting a gradually applied (static) load; namely, unit-stress is proportional to unit-strain until the proportional limit is reached. Hence, the energy U stored in the bar when the unit-stress in the bar is s (which is equal to the work, w, done on the bar in causing the stress s, provided that s is not greater than the proportional limit) may be expressed as follows:

$$U = w = \tfrac{1}{2}Pe,$$

in which P is the final value of the gradually applied load and e is the total elongation of the bar. But since s is within the proportional limit we have $\frac{s}{\epsilon} = E$. Further, $s = \frac{P}{a}$ where a is the cross-sectional area of the bar. Hence, $U = \frac{1}{2}\frac{s^2}{E}al$.

Now, since the assumption is made that this relation between an energy load and the corresponding stress remains the same regardless of the way the energy is delivered to the bar, the relation between any energy load U that causes axial stress and the unit-stress s is

$$U = \frac{1}{2}\frac{s^2}{E}al \quad \text{or} \quad s = \sqrt{\frac{2UE}{al}}, \quad . \quad . \quad . \quad . \quad . \quad (201)$$

in which s must not exceed the proportional limit of the material. If U is expressed in inch-pounds, E in pounds per square inch, a in square inches, and l in inches, s will be expressed in pounds per square inch. Equation (201) shows that the energy absorbed per unit-volume of the material, when stressed to the proportional limit, s_e, is $\frac{1}{2}\frac{s_e^2}{E}$; this value is represented by the area OCD in Fig. 255 and is called the *modulus of resilience* of the material.

The ideal material, then, for resisting energy loads in service in which the material must not incur permanent distortion, is one having a high modulus of resilience, that is, one having a high proportional limit and a low modulus of elasticity.

Ultimate Energy Resistance of a Material. Toughness.—The ultimate energy resistance of a material is the maximum amount of work that can be done on the material per unit of volume, without causing the

---

stress. But with energy loads as they usually occur in engineering practice the assumption is probably justified although experimental verification is difficult to obtain.

material to rupture. It is represented by the total area under the stress-strain curve, area *OCEF* in Fig. 255. Most of this work is dissipated in heat and in causing permanent deformation of the material, and only a very small part is stored in the material as stress-energy that can be recovered when the stress is released. The property of a

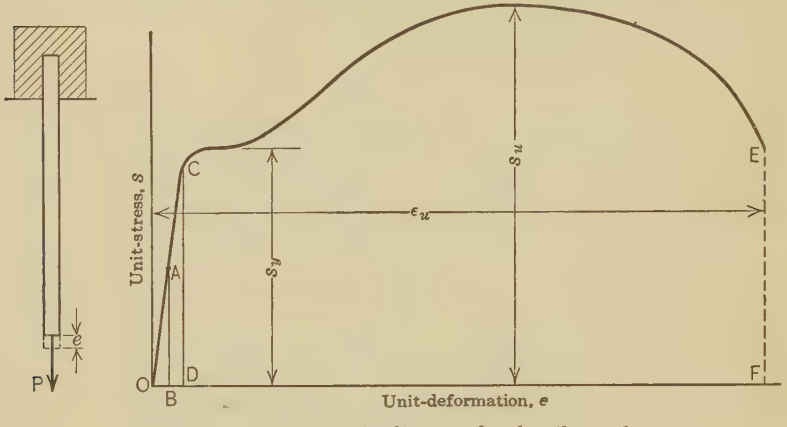

FIG. 255.—Stress-strain diagram for ductile steel.

body by virtue of which work can be done on the body when stressed beyond its elastic limit is called *toughness*. A tough material is needed, therefore, to resist energy loads when the material in service is likely to be stressed beyond its yield-point.

For materials having a stress-strain diagram similar to that shown in Fig. 255 (that is, for ductile steel) the area under the curve is given approximately by the expression $\frac{s_y + s_u}{2}\epsilon_u$, and hence the ultimate energy load U that the bar of constant cross-section can resist is, approximately,

$$U = \frac{s_y + s_u}{2}\epsilon_u \cdot al, \quad . \quad . \quad . \quad . \quad (202)$$

in which s_y and s_u are the yield-point and ultimate strength of the material, respectively, and ϵ_u is the ultimate unit-deformation. And, for material having approximately a parabolic stress-strain diagram (Fig. 256) such

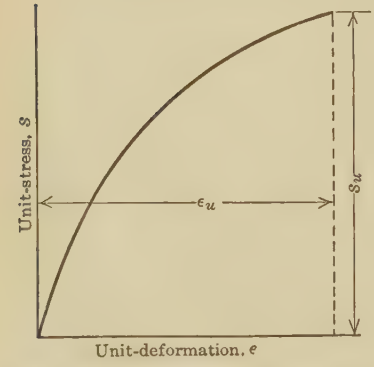

FIG. 256.—Stress-strain diagram for brittle material.

as cast iron and concrete, the area under the curve, is approximately $\frac{2}{3}s_u\epsilon_u$, and hence the ultimate energy load a bar of such a material can resist is, approximately,

$$U = \tfrac{2}{3}s_u\epsilon_u al. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (203)$$

101. Comparison of Effects of Static and Energy Loads.—The resistance offered by a bar to a *static* axial load, P, depends only on the maximum unit-stress developed, which occurs on the smallest cross-section ($P = as$), whereas the resistance of the bar to an energy load, U, as indicated by equation (201) depends not only on the maximum unit-stress, s, but also (1) on the distribution of the stress throughout the body, since the energy absorbed by a given unit of volume is $\frac{1}{2}\frac{s^2}{E}$, and hence depends on the degree to which that volume is stressed, and (2) on the number of units of volume (al) of material in the bar. The influence of these two factors may be shown as follows:

(1) The static strengths of the two cylindrical bars shown in Fig. 257 when subjected to axial loads are equal, since the smallest cross-sections are equal, and hence the loads P required to produce a given stress s in the bars are equal ($P = as$). The energy loads required to produce a given stress in the two bars, however, are very different; the bar having the constant diameter, and hence a uniform distribution of stress throughout the length of the bar, is able to absorb the greater amount of energy. For example, if the diameter of the upper half of the bar shown in Fig. 257(b) is reduced from $2d$ to d, the area a (and also volume al) will thereby be decreased to one-fourth of the original area (and volume), and hence the unit-stress will be increased to four times the original value. Therefore, the energy absorbed per unit volume $\left(\frac{1}{2}\frac{s^2}{E}\right)$ by this upper half will be sixteen times as great as that absorbed per unit-volume when the diameter is $2d$. Thus, in the expression $\frac{1}{2}\frac{s^2}{E}al$ for the upper half of the bar, the factor $\frac{1}{2}\frac{s^2}{E}$ has been increased more than the factor al has been de-

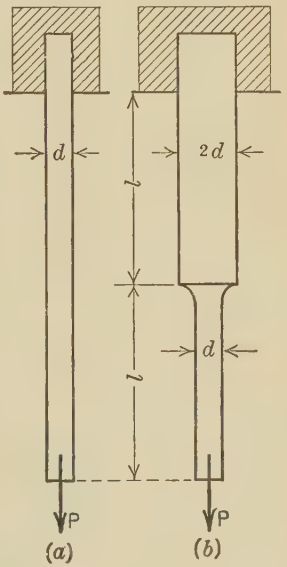

FIG. 257.—The bars have equal resistance to static loads, but not to energy loads.

creased; the total energy absorbed by the upper half of the bar being increased to four times its original value by reducing the diameter of the upper half from $2d$ to d.

This method of increasing the energy load that can be applied to a bar was brought forcibly to the attention of engineers in the early development of the steam engine by Professor John E. Sweet. The bolts in the connecting-rod head when made in the usual form with full-sized shanks and threaded ends as in Fig. 258(a) frequently broke [2] in service due to the energy load delivered by the reciprocating parts of the engine. By turning down the bolts so that the area of cross-section of the shank was equal to that at the roots of the threads, as shown in Fig. 258(b), the difficulty was removed since most of the energy of

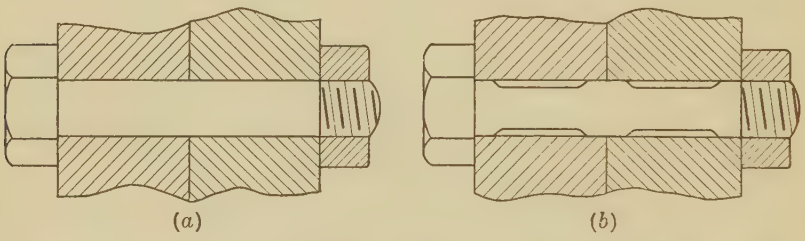

FIG. 258.—Energy resistance of bolt increased by decreasing diameter of shank.

the reciprocating parts was then absorbed in the shank of the bolt, and the amount absorbed at the roots of the thread was thereby greatly reduced with a corresponding reduction in the stress developed at the roots of the threads.

It will be seen, therefore, that uneven distribution of stress throughout a member may greatly reduce the energy load that can be applied to the body.

2. If the stress in a bar is distributed uniformly throughout the bar, that is, if the area of cross-section is constant, the energy load that the bar will resist may be increased by increasing the length of the bar since this increases the volume, each unit of volume absorbing the same amount $\left(\frac{1}{2}\frac{s^2}{E}\right)$ of energy.

This method of increasing the energy resistance of a member was

[2] As is shown in the preceding chapter, failure of material is frequently due to the repeated application of a stress if the stress is above a limiting minimum value called the endurance limit, and this was probably the cause of the failure of the bolts here referred to. But, by reducing the stress in the bolts, as stated, the repetitions of this lower stress did not cause failure.

clearly illustrated in the early development of rock drills.[3] The cylinder heads of the rock drills were attached to the cylinder as shown in Fig. 259(*a*), and since in the operation of the drill it is impossible to avoid occasional sharp blows of the piston on the cylinder head, the bolts were subjected to severe energy loads and, as a consequence, broke. The trouble was remedied by using bolts long enough to extend through both cover plates as shown in Fig. 259(*b*).

Effect of Localized Stress.—Sharp changes in section should be avoided in members subjected to energy loads, especially in members composed of brittle material. Even in ductile material the localized stresses at such abrupt changes of section cause very high values of

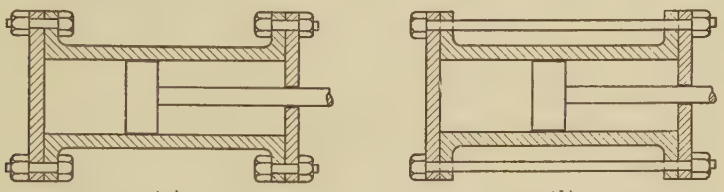

FIG. 259.—Energy resistance of bolts increased by making them longer.

energy to be absorbed per unit volume of the highly stressed material, which may cause the material to rupture before the stress (and hence energy) can be distributed to other portions of the member.

102. Special Cases of Axial Energy Loads.—(*a*) *Sudden Load.*—A sudden load is a force that remains constant throughout the entire deformation, e, of the resisting member. Thus, if a sudden axial load P acts on a bar the corresponding energy load, U, supplied to the bar (that is, the work done on the bar in causing the deformation e) is Pe.

Thus the expression for s in equation (201) becomes,

$$s = \sqrt{\frac{2PeE}{al}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204)$$

and since $\frac{e}{l} = \epsilon$, and $E\epsilon = s$, this expression may be written

$$s^2 = 2\frac{P}{a}s \quad \text{or} \quad s = 2\frac{P}{a}. \quad . \quad . \quad . \quad . \quad . \quad (205)$$

But if the load P were a static (gradually applied) load, $\frac{P}{a}$ would be

[3] Halsey, F. A., "Materials and Constructions for Resisting Shock," American Machinist, Sept. 9, 1915, p. 459.

the unit-stress developed. Therefore, within the proportional limit, the unit-stress caused by a sudden load is twice as great as that caused by the same load when gradually applied.

(*b*) *Energy Load Due to a Falling Body.*—If the energy load is delivered to a bar by a body of weight W falling through a height h before it comes in contact with the bar (see Fig. 263), the value of U in equation (201) is $W(h + e)$, where e is the total elongation of the bar. Hence,

$$s = \sqrt{\frac{2W(h + e)E}{al}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

But since e depends on s it will be convenient to express e in terms of s, and at the same time to express s in terms of the unit-stress, s_1, and deformation, e_1, that would be caused by the same weight if gradually applied. Thus, the above equation may be written:

$$s^2 = \frac{2WE(h + e)}{al}.$$

But $\frac{W}{a}$ equals the unit-stress, s_1, due to the load W when gradually applied. Further, within the proportional limit, $s = E\epsilon$; also $e = \epsilon l$, and hence $\frac{E}{l} = \frac{s}{e}$. Therefore the above equation may be written:

$$s = 2s_1\left(\frac{h}{e} + 1\right).$$

Now the ratio of unit-stress to unit-deformation is assumed to be constant within the proportional limit regardless of the kind of load that causes the stress and deformation. Hence,

$$\frac{s}{\epsilon} = \frac{s_1}{\epsilon_1},$$

$$\text{Also, } \frac{s}{e} = \frac{s_1}{e_1}, \text{ since } e = \epsilon l \text{ and } e_1 = \epsilon_1 l,$$

in which s_1 is the unit-stress and e_1 the total deformation that would be caused by a static load W. Thus $e = \frac{se_1}{s_1}$ and hence

$$s = 2s_1\left(\frac{hs_1}{se_1} + 1\right).$$

Whence, the maximum stress is,

$$s = s_1 + s_1\sqrt{1 + \frac{2h}{e_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207)$$

in which s must not exceed the proportional limit. This equation shows that the stress due to an energy load caused by a falling weight may be greatly in excess of that caused by the same weight when gradually applied.

Similarly, since $\frac{s}{E} = \epsilon$ and $\frac{s_1}{E} = \epsilon_1$, the elongation e of the bar due to the falling weight is

$$e = e_1 + e_1\sqrt{1 + \frac{2h}{e_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

in which e_1 is the elongation of the bar that would be caused by the same weight if gradually applied.

103. Working Stress and Working Value of Energy.—If the equivalent-static-load method (see Art. 98) is used in making allowance for the effect of the impact of the load, the working stress may be the same with impact loading as with static loading. Specifications usually require, however, slightly better material (more uniform in quality and tougher) for use in structures subjected to impact, such as, ships, locomotives frames and bridges, than for structures subjected to loads that are approximately static loads, such as buildings.

If, on the other hand, the load is considered to be an energy load, the working stress, or rather the working value of the energy, should be based on the modulus of resilience $\left(\frac{1}{2}\frac{s_e^2}{E}\right)$ of the material, particularly if the maximum usable energy of the material is the elastic energy, that is, if the resisting member would be structurally damaged if stressed above the proportional limit (or yield point) of the material. But the toughness of the material should also be considered in order to insure reserve energy capacity for resisting overloads and high localized stresses.

A working value of energy should, of course, be considerably less than the modulus of resilience; the working value chosen depends on the way the energy load is applied, on the service rendered by the member or machine, on the toughness of the material (its capacity to resist overloads, etc.), and on the form of the member with reference particularly to the way the energy is distributed throughout the member. In well-designed machines (in which the energy load on any member is reduced to a minimum by the balancing of moving parts, by the use of springs, by adjusting the relative stiffness of the component parts and of the connected members, etc.), the working value of the energy may be made one-fifth to one-tenth of the modulus of resilience. It should be noted that the resilience varies as the square of the stress developed (assuming E to be constant as in the case of steel), and hence when the

energy absorbed by the member is reduced a given amount the corresponding stress is reduced considerably less than this amount.

Table 7 should be helpful in selecting material for resisting energy loads and in selecting working values of energy to use in proportioning members subjected to energy loads.

TABLE 7

AVERAGE VALUES OF MODULUS OF RESILIENCE AND TOUGHNESS

Material	Tensile Proportional Limit (lb. per sq. in.) s_e	Tensile Ultimate Strength (lb. per sq. in.) s_u	Tensile Modulus of Elasticity (lb. per sq. in.) E	Ultimate Elongation per Inch of Length (in.) ϵ_u	Tensile Modulus of Resilience (in.-lb. per cu. in.) $\frac{1}{2}\frac{s_e^2}{E}$	Toughness in Tension (Represented by Area under Stress-strain Diagram) (in.-lb. per cu. in.)
Low carbon steel.....	30,000	60,000	30,000,000	0.35	15.0	15,700
Medium carbon steel..	45,000	85,000	30,000,000	0.25	33.7	16,300
High carbon steel.....	75,000	120,000	30,000,000	0.08	94.0	5,100
Special alloy steel (Heat treated).....	200,000	230,000	30,000,000	0.12	667.0	22,000
Gray cast iron.....	6,000	20,000	15,000,000	0.005	1.2	70
Malleable cast iron....	20,000	50,000	23,000,000	0.10	17.4	3,800
Rolled bronze.....	40,000	65,000	14,000,000	0.30	57.2	15,500
Timber (Hickory).....	5,500*	10,000*	2,400,000*	.....	6.32*	.....
Duralumin (tempered)	18,500	51,200	10,000,000	0.30	17.1	.....

* In compression. The effectiveness of timber for resisting energy loads is greater than that indicated by the value of the resilience given in the table, due in part to the fact that timber may be stressed somewhat above its proportional limit without destroying its usefulness; that is, the properties of resilience and toughness are not sharply defined.

ILLUSTRATIVE PROBLEMS

Problem 403.—Fig. 260 shows two bolts with square threads; they have the same dimensions except that one has the shank turned down to a diameter equal to that at the root of the threads for a length of 10 in. If both bolts are made of soft steel

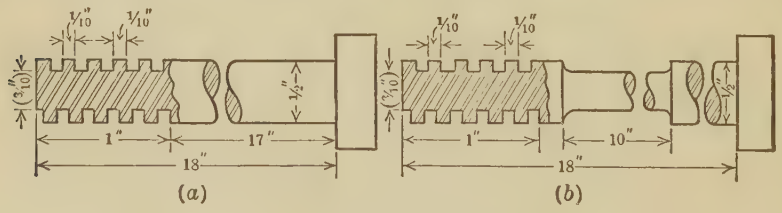

FIG. 260.

with a tensile proportional limit of 32,000 lb. per sq. in., find the axial static load and the axial energy load that each bolt will resist when stressed to the proportional limit.

Solution.—Static Loads.—The static loads P that the two bolts can resist are equal since the least cross-sectional areas are equal. Thus,

$$P = as = \frac{\pi(0.3)^2}{4} 32{,}000 = 2{,}265 \text{ lb.}$$

Energy Loads.—The cross-sectional area, a_1, of each bolt at the root of the threads is 0.0706 sq. in., and area a_2 of the shank is 0.1963 sq. in. Hence when the unit-stress s_1 in the first inch of the bolt (the threads are neglected) is 32,000 lb. per sq. in., the unit-stress s_2 in the shank of the bolt is only

$$\frac{0.0706}{0.1963} \times 32{,}000 = 11{,}520 \text{ lb. per sq. in.}$$

Therefore, the energy load U that can be absorbed by the bolt in Fig. 260(a) when stressed to 32,000 lb. per sq. in. is

$$U = \frac{1}{2}\frac{s_1^2}{E} a_1 l_1 + \frac{1}{2}\frac{s_2^2}{E} a_2 l_2$$

$$= \frac{1}{2}\frac{(32{,}000)^2}{30{,}000{,}000} 0.0706 \times 1 + \frac{1}{2}\frac{(11{,}520)^2}{30{,}000{,}000} \times 0.1963 \times 17$$

$$= 1.21 + 7.38 = 8.59 \text{ in.-lb. or } 0.715 \text{ ft.-lb.}$$

Likewise, the maximum energy load that can be applied to the bolt in Fig. 260(b) without causing a stress greater than the proportional limit is

$$U = \frac{1}{2}\frac{s_1^2}{E} a_1 l_1 + \frac{1}{2}\frac{s_2^2}{E} a_2 l_2$$

$$= \frac{1}{2}\frac{(32{,}000)^2}{30{,}000{,}000} 0.0706 \times 11 + \frac{1}{2}\frac{(11{,}520)^2}{30{,}000{,}000} 0.1963 \times 7$$

$$= 13.3 + 3.05 = 16.4 \text{ in.-lb. or } 1.36 \text{ ft.-lb.}$$

Therefore, a body weighing only 0.715 lb. falling a distance of only 1 ft. would supply enough energy to the bolt in Fig. 260(a) to stress it to the proportional limit *provided all the energy were absorbed by the bolt*, whereas a body weighing 2265 lb. would be required to cause the same stress if its weight were gradually applied to the bolt. Further, turning down the shank of the bolt as shown in Fig. 260(b) nearly doubles the energy resistance of the bolt without affecting its static resistance.

The results found in this problem further emphasize the importance of a uniform distribution of stress throughout a body that is required to resist energy loads. The results also indicate that energy loads would be extremely serious if the energy did not distribute itself throughout the whole structure or machine, or if it did not dissipate itself in the ways stated in Art. 99. These facts are further emphasized in the next problem.

Problem 404.—In Fig. 261 is shown a punching machine. The flywheel A is made to rotate by means of friction discs (not shown); it supplies the energy required to punch a $\frac{7}{8}$-in. hole through a $\frac{5}{8}$-in. steel plate. The diameter of the screw at the roots of the threads is 4 in. and the length of the screw beneath the frame is 10 in.

(a) Find the maximum compressive unit-stress in the stem of the screw assuming

that the load is a static axial load. (*b*) Find the maximum unit-stress, assuming that the load is an energy load and that the screw absorbs all the energy. (*c*) Find same as under (*b*), assuming that the machine frame absorbs three-fourths of the energy and the screw the other fourth.

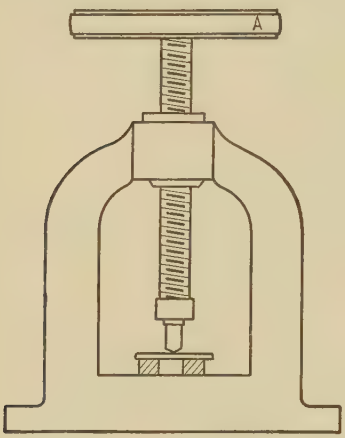

FIG. 261.

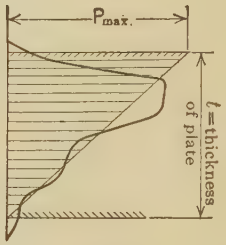

FIG. 262.

Solution.—(*a*) The maximum force P exerted on the punch, if the ultimate shearing unit-stress of the plate is 60,000 lb. per sq. in., is

$$P = \text{shearing area} \times 60{,}000$$

$$= \pi d \times t \times 60{,}000 = \pi \tfrac{7}{8} \times \tfrac{5}{8} \times 60{,}000 = 103{,}000 \text{ lb.}$$

And the unit-stress in the screw due to a static load of 103,000 lb. would be

$$s = \frac{P}{a} = \frac{103{,}000}{\pi(2)^2} = 8210 \text{ lb. per sq. in.}$$

(*b*) Tests show that the work diagram for the punching of steel plates is approximately that shown by the heavy line in Fig. 262, and that the area under the work diagram is equal, approximately, to the area under a triangular diagram (Fig. 262) in which the maximum pressure is that required to develop a shearing unit-stress of 60,000 lb. per sq. in. in the steel in order to punch the hole. Hence, the work done in punching the hole as represented by the area under the triangular diagram is

$$U = \frac{P}{2} \times t = \frac{103{,}000}{2} \times \frac{5}{8} = 32{,}200 \text{ in.-lb.}$$

If all of this work were stored in the part of the screw between the plate and the machine frame and none in the machine frame, the unit-stress in the screw would be

$$s = \sqrt{\frac{2UE}{al}} = \sqrt{\frac{2 \times 32{,}200 \times 30{,}000{,}000}{12.56 \times 10}}$$

$$= 124{,}000 \text{ lb. per sq. in.}$$

provided that this value is less than the proportional limit.

(*c*) But the frame is not rigid, and hence probably a considerable part of the work done by the plate on the punch and screw is stored in the frame of the machine. If three-fourths of the work is used in deforming the frame then the value for U in the above expression is 8050 in.-lb. and the unit-stress becomes

$$s = 62{,}000 \text{ lb. per sq. in.}$$

It is clear from these large values for s that a large part of the energy delivered to the screw is transmitted on to the machine frame. And, in general, the distribution of energy throughout the whole machine insures the safety of the machine; consideration of this fact is of great importance in the design of various types of machines and structures.

The results of this problem also suggest the desirability of a study of existing machines when designing members that are subjected to energy loads.

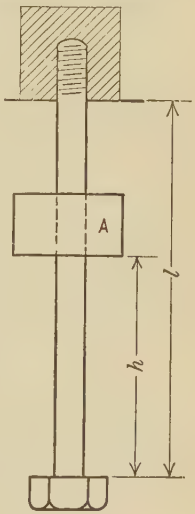

FIG. 263.

PROBLEMS

405. A body A, Fig. 263, having a weight of 20 lb. falls a height h of 4 ft. when it comes in contact with the end of a rod having a length l of 6 ft. (*a*) If the rod is made of soft steel having a proportional limit of 30,000 lb. per sq. in., what cross-sectional area should the rod have in order to prevent the unit-stress from exceeding the proportional limit? (*b*) If test specimens of the steel when tested had an ultimate strength of 60,000 lb. per sq. in. and an elongation of 30 per cent, estimate the weight of A required to rupture a rod having a diameter of 0.25 in. (NOTE.—All the energy of the falling weight is assumed to be absorbed by the rod.)

Ans. (*a*) $a = 0.890$ sq. in.; (*b*) $W = 685$ lb.

406. If the rod in Problem 405 were a stick of oak having a proportional limit of 3500 lb. per sq. in. and a modulus of elasticity of 1,800,000 lb. per sq. in., what should be the least cross-section of the stick to prevent the stress from exceeding the proportional limit? *Ans.* $a = 3.92$ sq. in.

407. A steel bar having a length of 10 in. and a cross-section of 1.0 sq. in. is struck axially by a moving body. The work done on the bar when the body is brought to rest is 6 ft.-lb. Find the maximum stress developed in the bar.

408. A structural steel bar and an oak stick used as direct tension members are required to absorb the energy of a falling weight. The cross-sectional area of the steel and oak bars are 2 and 3 sq. in., respectively; the lengths are 3 ft. for the steel bar and 4 ft. for the oak. The modulus of elasticity of the oak is 1/30 of that of steel. The proportional limit of the oak is 1/10 of that of steel. Which bar can resist the greater energy load without causing inelastic action in the bar?

409. The piston rods of the forging hammers in a certain steel plant frequently broke, and when they were finally replaced by a higher carbon steel the trouble ceased. If the proportional limit of the soft steel was 32,000 lb. per sq. in. and that of the higher carbon steel was 50,000 lb. per sq. in., how much more energy could the higher carbon steel rod absorb without being stressed above the proportional limit?

NOTE.—The breaking of the rods was probably due to the fact that the stress was repeated (repeated stress is discussed in the preceding chapter), but this was the secondary cause of the failure. (See footnote 2 of Art. 101.)

410. Fig. 264 shows stress-strain diagrams for medium carbon steel (about 0.30 per cent carbon) and for high carbon steel (about 0.80 per cent carbon). (*a*) If a bolt of medium carbon steel is stressed just to its proportional limit by an energy

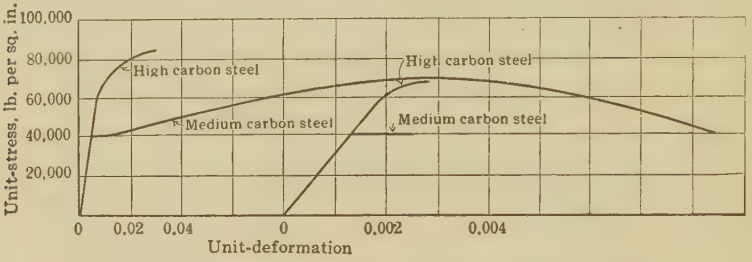

FIG. 264.—Stress-strain diagrams of two grades of steel.

load of 25 foot-pounds, how large an energy load will a bolt of the same size of the 0.80 per cent carbon steel resist when stressed to its proportional limit? (*b*) What will be the ratio of the energy loads that will cause rupture of the two bolts?

Ans. (*a*) 56.2 ft. lb.; (*b*) 6.5 approx.

411. In Fig. 265 are shown stress-strain diagrams for cast iron and for steel castings having about 0.20 per cent of carbon. Owing to the greater strength of steel the cross-sectional area of a steel casting for railway service (car couplers for

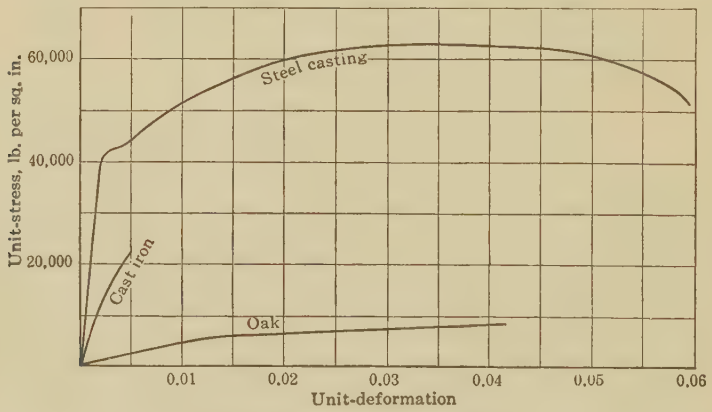

FIG. 265.—Stress-strain diagrams.

example) is made only 0.4 as great as that of a cast-iron casting. What will be the relative resistance to rupture, when subjected to an axial energy load, of the steel coupler and the cast-iron coupler if the lengths are the same? Since the service of a car coupler is not destroyed by permanent deformations and since in railway service such deformations are very likely to occur, the ultimate rather than the elastic energy loads are the governing factors. *Ans.* 18.4 times as great.

412. A tension member in a certain machine is subjected to an energy load. If the member can be given a permanent deformation without destroying its usefulness and can be made of either cast iron or oak, which material should be used provided that the strength of the member is the only factor to be considered (make use of Fig. 265)? *Ans.* Oak.

413. A bar 3 ft. in length and $\frac{1}{2}$ sq. in. in cross-sectional area is made of medium carbon steel (see Table 7) and is stressed to its proportional limit by an axial energy load U. If the diameter of the bar is turned down to one-half of its original value over one-fourth of its length, what is the value of the energy load that will stress the bar to its proportional limit?

104. Stresses in Beams Due to Energy Loads.—The amount of energy that a beam will absorb when stressed within the proportional limit of the material depends on the type of beam (cantilever, simple, fixed, etc.) and on the type of load (concentrated, distributed, etc.) in addition to the form and dimensions of the beam.

Simple Beam with Concentrated Load at Center.—Let it be required to find the maximum stress caused by an energy load U applied at

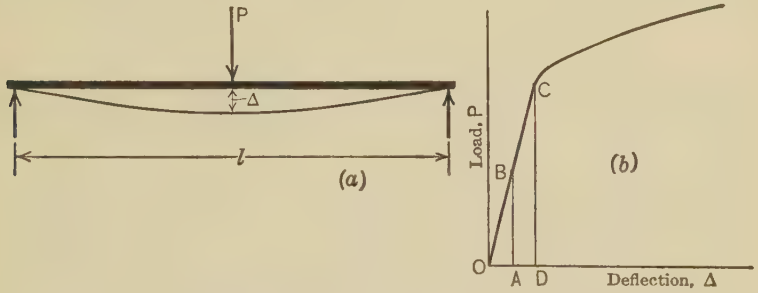

FIG. 266.—Energy resistance of beam; load at center.

the center of a simple beam, when the stress in the beam does not exceed the proportional limit of the material. Now the energy that can be stored in a beam is assumed to be the same when subjected to a gradually applied load as when subjected to an energy load due to a moving body. Thus the expression for the energy load U that can be absorbed by the beam when the stress does not exceed the proportional limit may be found as follows:

Let a simple beam having a span length of l feet be loaded with a gradually applied concentrated load at the mid-span, the final value of which is P pounds. The relation between the load, P, and the deflection, Δ, of the beam is shown by the curve in Fig. 266(*b*). The energy load U that can be absorbed by the beam (which is equal to the work done on the beam as a gradually applied load increases from

zero to the value P represented by AB in Fig. 266*b*) is represented by the area OBA, and may be found from the expression,

$$U = \tfrac{1}{2}P\Delta.$$

But from Art. 33 and 45 we have

$$\frac{Pl}{4} = \frac{sI}{c} \quad \text{and} \quad \Delta = \frac{1}{48}\frac{Pl^3}{EI}.$$

Therefore

$$U = \frac{1}{6}\frac{I}{c^2}\frac{s^2}{E}l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

But $I = ak^2$ (Art. 169) in which k is the radius of gyration of the cross-sectional area with respect to the neutral axis. Hence

$$U = \frac{1}{6}\frac{k^2}{c^2}\frac{s^2}{E}al \quad \text{or} \quad s = \frac{c\sqrt{6}}{k}\cdot\sqrt{\frac{UE}{al}}. \quad . \quad . \quad . \quad . \quad (210)$$

If U is expressed in inch-pounds, E in pounds per square inch, a in square inches, and k, c, and l in inches, s will be expressed in pounds per square inch.

Since the numerical value of the ratio $\dfrac{k^2}{c^2}$ is always the same for similarly shaped sections, the amount of energy that a horizontal rectangular beam will absorb when the short dimension is vertical will be the same as that absorbed when the long dimension is vertical.

PROBLEMS

414. Show that if the beam has a rectangular cross-section, equation (210) reduces to

$$U = \frac{1}{18}\frac{s^2}{E}al = \frac{1}{9}\left(\frac{1}{2}\frac{s^2}{E}\right)al, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

which shows that the energy load which this beam can resist is only one-ninth as large as the energy load it could resist if it were used as a tension member and stressed to the same maximum value s.

415. Show that if the beam has a circular cross-section, equation (210) reduces to

$$U = \frac{1}{24}\frac{s^2}{E}al = \frac{1}{12}\left(\frac{1}{2}\frac{s^2}{E}\right)al. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

Simple Beam with Uniformly Distributed Load. — The relation between a gradually applied load and the deflection would be similar to that shown in Fig. 266(*b*). The final values of the load and maximum deflection are W and Δ, respectively. The deflection at any section is y and the load that acts through this distance is wdx (Fig. 267). Hence the energy load U that can be applied (which is equal to the work done on the beam) is

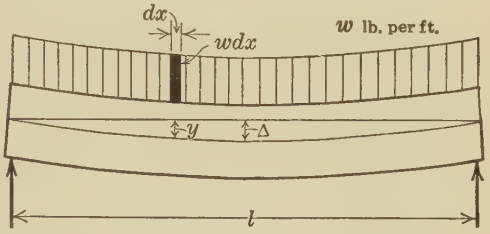

FIG. 267.—Energy resistance of beam; uniform load.

$$U = \int_0^l (\tfrac{1}{2} wdx \cdot y),$$

and by making use of the expression for y found in Art. 44 we have,

$$U = \frac{1}{2}\int_0^l wdx \frac{w}{EI}\left(\frac{l^3 x}{24} - \frac{lx^3}{12} + \frac{x^4}{24}\right)$$

$$= \frac{1}{2}\frac{w^2}{EI}\left(\frac{l^5}{48} - \frac{l^5}{48} + \frac{l^5}{120}\right) = \frac{1}{240}\frac{w^2 l^5}{EI}. \quad . \quad . \quad (213)$$

But w may be expressed in terms of the maximum unit-stress in the beam by means of the flexure formula, Art. 31; namely, $\frac{1}{8} wl^2 = \frac{sI}{c}$. Therefore

$$U = \frac{1}{240}\left(\frac{8sI}{cl^2}\right)^2 \frac{l^5}{EI} = \frac{4}{15}\frac{I}{c^2}\frac{s^2}{E} \cdot l = \frac{4}{15}\frac{k^2}{c^2}\frac{s^2}{E} al. \quad . \quad . \quad (214)$$

PROBLEMS

416. Show that if the beam has a rectangular cross-sectional area, equation (214) reduces to

$$U = \frac{4}{45}\frac{s^2}{E} al = \frac{8}{45}\left(\frac{1}{2}\frac{s^2}{E}\right) al, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

which shows that the energy load which this beam can resist is only $\frac{8}{45}$ as large as the energy load it could resist if it were used as a tension member and subjected to the same maximum unit-stress.

417. Show that for a beam having a circular cross-sectional area equation (214) reduces to

$$U = \frac{1}{15}\frac{s^2}{E} al = \frac{2}{15}\left(\frac{1}{2}\frac{s^2}{E}\right) al. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

105. Effect of Form on Energy Resistance of Beams.—Equations (210) and (214) and the equations given in Problems 414 to 417 show that the material in a beam having a *constant* cross-section is inefficient in absorbing energy. For example, the expression in Problem 414 shows that a rectangular beam, when loaded at the mid-span with a concentrated load, can absorb only one-ninth as much energy as the same beam could absorb if all the material in the beam were stressed to the same degree.

The fact that the material in a beam having a constant cross-section is not effective in resisting an energy load arises from two causes: (1) since the bending moment is relatively small at sections near the supports (for simple beams), the unit-stress, even in the outer fibers at these sections, is necessarily small, and hence the material toward the ends of simple beams can absorb very little energy; and (2) the material near the neutral plane throughout the length of the whole beam can develop only small stresses and hence can absorb only a small amount of energy per unit volume $\left(\frac{1}{2}\frac{s^2}{E}\right)$.

The first of these difficulties is largely overcome in leaf springs by making the leaves of gradually decreasing length (Fig. 268) so that the

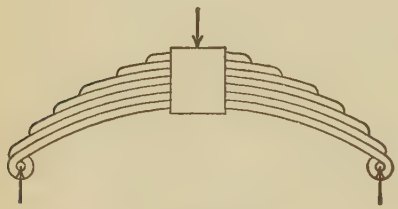

FIG. 268.

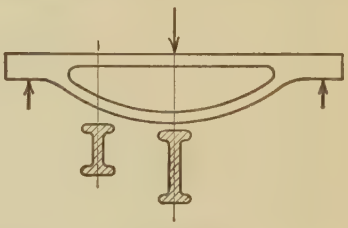

FIG. 269.

moment of inertia of the cross-sections decreases towards the supports approximately in proportion to the bending moment. Incidentally it is well to note that a leaf spring absorbs a considerable amount of energy due to friction between the leaves.

The second of the above difficulties may be overcome partly by placing as much of the material as far from the neutral surface as is practicable, as for example, in the form of an I-section. If now a simple beam with an I-section is also made so that the cross-sectional area decreases towards the supports, as is sometimes done in forged axles, etc. (Fig. 269), the resistance of the beam to energy loads is very much greater than that of a beam having a constant cross-section that would have the same static strength as the beam with the variable cross-section.

It is clear, therefore, that when a beam is subjected to an energy load the distribution of the stress in the beam should be as uniform as possible throughout the beam so that the energies absorbed by all unit volumes will be approximately equal.

Again, the energy absorbed by a beam as shown by equations (210) and (214) increases with the volume of the beam provided the other influencing factors discussed above are constant. Thus, if a simple beam has a constant cross-section, the amount of energy it will absorb *increases* as the length of the span of the beam is increased, whereas the static strength *decreases* with an increase in the length of the span.

106. Special Cases of Energy Loads on Beams.—*Energy Load Due to a Falling Weight.*—It is convenient frequently to express the unit-stress caused by an energy load, due to a falling weight W, in terms of the unit-stress that would be caused by the force W when gradually applied. Similarly, the deflection caused by a falling weight W may be expressed in terms of the deflection that would be caused by a load W when gradually applied. This may be done by substituting in equation (210) or (214) the value of U for the beam in question similar to the method used in Art. 102; or it may be done as follows:

If a body having a weight W falls from a height h on a simply supported beam, the maximum deflection, Δ, of the beam will be proportional to the maximum unit-stress, s, developed, and the ratio of s to Δ is assumed to be the same, within the proportional limit, as that of the unit-stress, s_1, to the deflection, Δ_1, caused by a static load equal to W, thus

$$\frac{\Delta}{\Delta_1} = \frac{s}{s_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

provided the proportional limit of the material is not exceeded.

Further, if Q is a static load that causes a deflection, Δ, equal to that caused by the energy load, the work done by Q, which is $\frac{1}{2}Q\Delta$, will be equal to the energy supplied or given up by the falling body, and since the assumption is here made that all the energy of the falling body, $W(h + \Delta)$, is absorbed in stressing the beam, then $\frac{1}{2}Q\Delta$ will be equal to the energy absorbed by the beam. Hence

$$W(h + \Delta) = \tfrac{1}{2}Q\Delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

But the static loads are proportional to the stresses they develop and hence we have

$$\frac{\Delta}{\Delta_1} = \frac{s}{s_1} = \frac{Q}{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

By combining (218) and (219) there is found

$$s = s_1 + s_1\sqrt{1 + \frac{2h}{\Delta_1}} \quad \text{and} \quad \Delta = \Delta_1 + \Delta_1\sqrt{1 + \frac{2h}{\Delta_1}}, \quad . \quad (220)$$

in which s and Δ are the unit-stress and deflection, respectively, due to the falling body, and s_1 and Δ_1 are the stress and deflection, respectively, due to a static load equal to the weight of the falling body. Expressions for s_1 and Δ_1 are given in Chapters IV and V. Equations (220) show that a body which causes a relatively small stress and deflection when applied as a static load may cause a large stress and deflection if allowed to drop on the beam through a relatively short distance.

Sudden Load.—If the value of h in equation (220) is zero, that is, if the load W is a sudden load (see Art. 102 for definition of sudden load), then the values of s and Δ due to the sudden load are

$$s = 2s_1 \quad \text{and} \quad \Delta = 2\Delta_1. \quad . \quad . \quad . \quad . \quad . \quad (221)$$

Therefore, a sudden load applied to a beam will cause twice the stress and twice the deflection that will be caused by the same load when gradually applied.

ILLUSTRATIVE PROBLEM

Problem 418.—The proportional limit for hickory may be taken at 3000 lb. per sq. in., and its modulus of elasticity at 1,500,000 lb. per sq. in. (*a*) Will a weight of 20 lb. falling 6 in. on the center of the span of a hickory beam 4 in. square cause a unit-stress above the proportional limit, if the beam has a span of 3 ft.? (Assume that the supports of the beam are rigid and that all the energy delivered by the falling weight is absorbed by the beam.) (*b*) How many inches will the beam deflect? (*c*) What should be the length of the span to make the stress equal 3000 lb. per sq. in.?

Solution.—(*a*) *First Method.*—The deflection of the beam will be neglected in comparison with 6 in. and hence the energy U delivered to the beam is

$$U = Wh = 20 \times 6 = 120 \text{ in.-lb.}$$

The unit-stress caused by this energy load is (see Prob. 414).

$$s = \sqrt{\frac{18UE}{al}} = \sqrt{\frac{18 \times 120 \times 1{,}500{,}000}{4 \times 4 \times 36}} = 2370 \text{ lb. per sq. in.}$$

(*a*) *Second Method.*—If the deflection of the beam is not neglected the maximum unit-stress s developed in the beam is

$$s = s_1 + s_1\sqrt{1 + \frac{2h}{\Delta_1}},$$

The unit-stress s_1 due to a static load of 20 lb. at the center of span is

$$s_1 = \frac{Mc}{I} = \frac{1}{4}Pl \cdot \frac{c}{I} = \frac{1}{4} \times 20 \times 36 \times \frac{2}{\frac{1}{12}(4)^4} = 16.9 \text{ lb. per sq. in.}$$

The maximum deflection, Δ_1, of the beam caused by the 20-lb. static load is

$$\Delta_1 = \frac{1}{48}\frac{Pl^3}{EI} = \frac{1}{48}\frac{20 \times (36)^3}{1{,}500{,}000 \times \frac{1}{12}(4)^4} = 0.000608 \text{ in.}$$

Hence,

$$s = 16.9 + 16.9\sqrt{1 + \frac{2 \times 6}{0.000608}}$$

$$= 16.9 + 2365 = 2382 \text{ lb. per sq. in.}$$

and hence the error introduced in the first method of solution by neglecting the deflection of the beam is very small.

(*b*) The maximum deflection of the beam is

$$\Delta = \Delta_1 + \Delta_1\sqrt{1 + \frac{2h}{\Delta_1}} = 0.000608 + 0.000608 \times 140$$

$$= 0.000608 + 0.0853 = 0.0859 \text{ in.}$$

(*c*) Since the deflection may be neglected in determining the unit-stress, we have

$$s = \sqrt{\frac{18UE}{al}} = 3000.$$

Hence

$$l = \frac{18UE}{a \times (3000)^2} = \frac{18 \times 120 \times 1{,}500{,}000}{16 \times (3000)^2} = 22.5 \text{ in.}$$

PROBLEMS

419. Three steel beams, *A*, *B* and *C*, all of the same dimensions, are arranged as shown in Fig. 270 similar to one type of draft gear. Each beam is 3 in. square and has a span of 12 in. (*a*) What static load applied at the mid-span will cause a maximum unit-stress of 20,000 lb. per sq. in.? (*b*) Will the beams when arranged as shown in Fig. 270 resist a greater static load than any one of the beams? (*c*) What is the weight of a body which, when allowed to fall a height h of 8 in. will cause a stress of 20,000 lb. per sq. in.? (*d*) What stress would the weight in (*c*) cause if it were resisted by only one of the three beams? *Ans.* (*a*) 30,000 lb.; (*b*) no; (*c*) 30 lb.; (*d*) 34,700 lb. per sq. in.

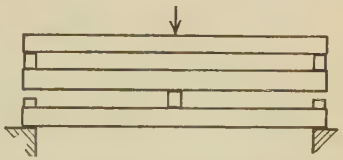

FIG. 270.—Arrangement of beams for absorbing energy.

420. A steel bar 2 in. in diameter and 3 ft. long remains in a horizontal position as it falls a distance of 4 ft. and strikes rigid supports at its ends. If all the energy is absorbed by the beam what unit-stress is developed in the beam, assuming that the proportional limit is not exceeded, and that steel weighs 490 lb./ft.^{3}

421. A high carbon steel shaft has a diameter of 4 in. It is used as a simple beam with a span of 6 ft. The proportional limit of the material is 60,000 lb. per sq. in. If a moving body strikes the beam at mid-span and two-thirds of the energy of the moving body is absorbed by the beam, how much energy does the body possess if it produces a unit-stress equal to one-half of the proportional limit? *Ans.* 1690 in.-lb.

107. Torsional Energy Load.—If one end of a solid cylindrical shaft of length l, radius c and area of cross-section a is fixed at one end and a twisting moment is applied to the other end by means of a moving body (such as a flywheel connected to the shaft by a clutch), the maximum shearing stress developed in the shaft in terms of the energy absorbed by the shaft may be found as follows:

The elastic work done on the shaft by a gradually applied twisting moment, whose final value is T, is $\frac{1}{2}T\theta$ where θ is the angle of twist (in radians) of the free end of the shaft. Assuming as in the preceding articles that the relation between stress and strain is the same under impact as under static load, the energy U absorbed by the shaft when stressed within the proportional limit is, referring to Chap. III,

$$U = \frac{1}{2}T\theta = \frac{1}{2}\frac{T^2 l}{E_s J} = \frac{{s_s}^2 J l}{2c^2 E_s} = \frac{{s_s}^2 \pi c^2 l}{4E_s} = \frac{{s_s}^2}{4E_s} al$$

Hence

$$s_s = 2\sqrt{\frac{UE_s}{al}}. \quad \ldots\ldots\ldots \quad (222)$$

PROBLEM

422. If a round structural steel bar having a given value of a and l is to be used to absorb energy without acquiring plastic deformation, would it be better to use the bar as a torsion member or as an axially loaded tension member? (Consult Table 1, Chap. I.)

108. Reduction in Energy Load Due to Inertia of Resisting Member.—It was stated in Art. 99 that the value of the energy load U as used in this chapter is the energy that is actually absorbed by the resisting member, and that one of the ways in which the energy of the moving body is dissipated is in causing local yielding of the resisting member. The energy dissipated in this way may be relatively large if the weight W of the resisting member is large compared to the weight P of the moving body. If, then, a third body of weight W_1 is attached to the resisting member, the energy load U applied to the member will thereby be decreased.

This principle is recognized in the design of structures and machines by reducing the allowance made for impact as the ratios W/P and

W_1/P increase. In accordance with this principle a heavy floor on a suspension bridge decreases the stresses, due to impact, in the vertical rods that connect the floor to the cables; for the same reason the specifications for the drop test for steel rails specify not only the weight of the striking tup but also that of the spring-supported anvil block to which the rail supports are attached, in addition to the height of drop; etc.

By using the principle of conservation of momentum as applied to inelastic bodies, an expression for U may be found in terms of W/P, W_1/P and E_k, where E_k is the energy of the moving body as it comes in contact with the resisting body; the expression is $U = nE_k$ in which n has the following values:

(*a*) For an axial energy load on a bar (Art. 100);

$$n = \frac{1 + q_1 + \frac{1}{3}q}{(1 + q_1 + \frac{1}{2}q)^2}, \text{ where } q_1 = \frac{W_1}{P} \text{ and } q = \frac{W}{P}.$$

(*b*) For an energy load applied at the center of a simple beam (Art. 104), and with W_1 attached at the center of the span;

$$n = \frac{1 + q_1 + \frac{17}{35}q}{(1 + q_1 + \frac{5}{8}q)^2}, \text{ where } q_1 = \frac{W_1}{P} \text{ and } q = \frac{W}{P}.$$

The values of n found from the above expressions should be regarded as relative rather than absolute values because of some of the assumptions made in the derivations. Waves of stress or vibrations caused by impact are here neglected.

REVIEW QUESTIONS AND PROBLEMS

423. Underline the correct property of the material named in the following statement. If a member in service must absorb energy due to collision with moving bodies without acquiring appreciable permanent deformation, the property of the material which best measures its resistance is the elastic limit, yield point, endurance limit, modulus of resilience, per cent elongation, ultimate strength.

424. Supply the missing word in the following statement. The total area under the stress-strain diagram of a material is a measure of the ———— of the material.

425. Is the following statement correct? The toughness of a material is a measure of the ability of the material to absorb energy without rupturing, especially when the material is used in a member free from abrupt changes of section so that localized stresses are not of importance.

426. Is the following statement correct? The elastic resistance of a member to *static loads* is not decreased by having a large part of the member subjected to stress less than the maximum stress in the member, whereas the elastic resistance of a member to energy loads is often greatly increased by decreasing the amount of material

in the member in such a way as to make the stress in the member more nearly uniformly distributed throughout the volume of the member.

427. A structural steel bar 20 in. long and an aluminum-alloy (duralumin) bar, each 1 in. in diameter, are subjected to equal energy loads causing axial stress. If it is specified that the energy absorbed by each bar per unit-volume is $\frac{1}{2}$ of the modulus of resilience of the material, what should be the length of the duralumin bar? (Consult Table 1 of Chap. I and Table 7 of this chapter.) *Ans.* 17.6 in.

428. A simple horizontal beam of span l having a rectangular section b by d is subjected to an energy load by dropping a weight at the center of the span. Show that the elastic energy resistance of the beam will be the same whether the longer dimension of the cross-section is placed vertically or horizontally.

429. A cantilever beam is subjected to an energy load by dropping a weight on the free end of the beam. The beam has a circular cross-section 2 in. in diameter for one-half of its length and 1 in. in diameter for the other half. Will the elastic energy resistance of the beam be greater with the small end as the free end, or with the large end as the free end? Explain.

Ans. Greater with the large end as the free end.

PART II. SPECIAL TOPICS

Several topics are discussed in Part II that lie just beyond those included in most of the undergraduate courses in this subject. Each chapter in Part II is independent of the other chapters in this part, with the exception of Chap. XIII, in which the theorems developed in Chap. XII are used. (Chaps. XII and XIII should be considered as a unit.) After completing the main portions of Part I, therefore, any one or more of the chapters in Part II may be studied without reference to the preceding chapters in this part.

The increasing use of analytical methods, in contrast with empirical rules, in solving the engineering problems that are continually arising in engineering industry has, in recent years, created a need for further training in the analysis of stresses and strains in various members of engineering structures and machines. There is likewise a need for a better understanding of the significance of calculated stresses in relation to the usable resistance of a member subjected to different types of loading. The topics discussed in Part II are presented with the aim of helping to satisfy these needs.

CHAPTER XII

DEFLECTION OF STATICALLY DETERMINATE BEAMS

(Moment-area Method)

Note.—Arts. 42 to 44 should be studied before this chapter is read.

109. Introduction.—In Chapters V and VI the elastic curve equation $M = EI\frac{d^2y}{dx^2}$ was used (1) to find the deflection of beams and (2) to find the reactions (both forces and couples) of beams for which the number of unknown reactions was greater than the number of equilibrium equations, that is, of statically indeterminate beams. The method of treating the equation in each case was to express M in terms of the reactions and x, and then to integrate both sides of the equation, determining constants in the equations (whether constants of integration or reactions) from the physical conditions satisfied by the beam. This method was called the double-integration method, and was the only method in use until about 1868.

Another method of treating the equation leads to the use of moments of areas of the moment-diagram $\left(\text{or of the } \frac{M}{EI}\text{-diagram}\right)$ for finding deflections of beams and reactions of statically indeterminate beams, and hence the method is called the *moment-area* method. For various purposes and for certain types of beams, as for example beams having a variable moment of inertia, it possesses advantages over the double integration method although it should, perhaps, be considered supplementary to, rather than a substitute for, the double integration method.

Two methods of using moments of areas of the moment-diagram $\left(\text{or of the } \frac{M}{EI}\text{-diagram}\right)$ will be discussed briefly; namely, the *slope-deviation* method and the *conjugate beam* method.[1] These two methods wlil be used in this chapter for determining the deflections of statically determinate beams, and the same two methods will be used in the following chapter for determining the reactions, as well as the deflections, of statically indeterminate beams.

[1] See "Deflections of Beams by the Conjugate Beam Method," by H. M. Westergaard, Journal of the Western Society of Engineers, vol. 26, Nov., 1921.

SLOPE-DEVIATION METHOD

110. Theorems of the Slope-deviation Method.—As explained in Art. 43, the equation of the elastic curve of a beam is $M = -EI\frac{d^2y}{dx^2}$, in which M is positive when it produces tensile stress on the bottom of a horizontal beam, and y is positive when measured downwards. This equation may be written

$$-\frac{M}{EI} = \frac{d\left(\frac{dy}{dx}\right)}{dx} \quad \text{or} \quad -\frac{M}{EI}dx = d\left(\frac{dy}{dx}\right), \quad . \quad . \quad (223)$$

in which $\frac{dy}{dx}$ is the slope of the elastic curve at a point for which the bending moment is M. If θ denotes the angle made by the tangent line to the elastic curve with the axis of the undeflected beam, then $\frac{dy}{dx} = \tan\theta$. But in a well designed beam θ is small, and hence, without appreciable error, $\tan\theta$ may be replaced by θ. Therefore,

$$d\theta = \frac{M}{EI}dx \quad \text{and} \quad \Delta\theta = \int\frac{M}{EI}dx, \quad . \quad . \quad . \quad . \quad (224)$$

in which $\Delta\theta$ denotes the change or increment in the angle θ; θ is positive when measured in the clockwise direction of rotation.

A graphical interpretation of this equation may be made as follows: Fig. 271(a) represents a beam subjected to a distributed load; Fig. 271(b) represents the $\frac{M}{EI}$ diagram for the beam, any ordinate in which is the bending moment at the section where the ordinate is erected divided by EI; if the beam is homogeneous and has a constant cross-section, EI is a constant and hence the $\frac{M}{EI}$-diagram has the same form as the moment-diagram. In Fig. 271(c) $PABQ$ represents the elastic curve of the beam. Now, as indicated in Fig. 271(b), $\frac{M}{EI}dx$ is represented by an elementary area under the $\frac{M}{EI}$-diagram, and $\int_A^B\frac{M}{EI}dx$ is represented by the total area under the $\frac{M}{EI}$-diagram between specified ordinates, such as ordinates erected at A' and B'. Further, $d\theta$ represents,

as shown in Fig. 271(*c*), the change in the slope of the elastic curve at two points a short distance, dx, apart, and hence $\int_{\theta_B}^{\theta_A} d\theta = \Delta\theta$ represents the change in slopes of the elastic curve at any two points, such as A and B. Therefore, equation (224) leads to the following theorem:

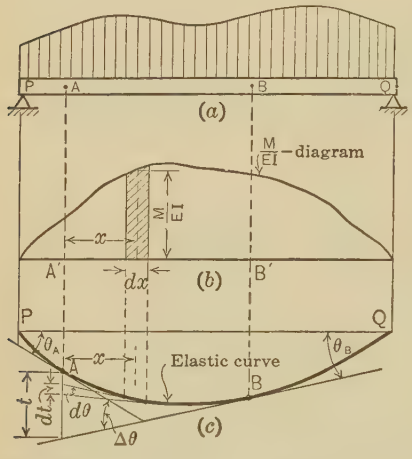

FIG. 271.—Deflection of a beam; slope-deviation method.

Theorem I.—When a straight beam is subjected to bending, the difference in the slopes of the elastic curve at any two points is equal in magnitude to the area of the $\frac{M}{EI}$*-diagram between the two points.*

Now let t denote the distance of any point A on the elastic curve, measured in a direction perpendicular to the original position of the beam, from a tangent drawn at any other point B on the elastic curve. The distance t will be called the *tangential deviation* of the point. Then,

$$t = \int dt = \int x d\theta, \quad \ldots\ldots \quad (225)$$

and from equation 224,

$$d\theta = \frac{M}{EI} dx.$$

Therefore,

$$t = \int \frac{Mx}{EI} dx. \quad \ldots\ldots\ldots \quad (226)$$

But, as is evident from Fig. 271(*b*), $\frac{Mx}{EI} dx$ is the moment of the elementary area, $\frac{M dx}{EI}$, of the $\frac{M}{EI}$-diagram about the ordinate through the point A whose tangential deviation is t, and hence $\int \frac{Mx}{EI} dx$ is the moment of that part of the $\frac{M}{EI}$-diagram between the two ordinates considered; the moment being taken about the ordinate through the point

whose tangential deviation is desired. The following theorem therefore may be stated:

Theorem II.—When a straight beam is subjected to bending, the distance of any point A on the elastic curve, measured normal to the original position of the beam, from a tangent drawn to the elastic curve at any other point B, is equal in magnitude to the moment of the area of the $\frac{M}{EI}$*-diagram between the two points about an ordinate through A.*

APPLICATIONS

111. Simple Beam; Concentrated Load at Mid-span.—The beam is shown in Fig. 272(*a*); the weight of the beam is assumed to be negligible. Fig. 272(*b*) shows the $\frac{M}{EI}$-diagram; the dangerous section is at the mid-span; the maximum bending moment is $\frac{Pl}{4}$, and the bending moment at a section the distance x from the left support is $\frac{P}{2}x$. Since the beam is assumed to be homogeneous and to have a constant cross-section, the $\frac{M}{EI}$-diagram has the same form as the moment-diagram. In Fig. 272(*c*) *ACB* represents the elastic curve of the beam.

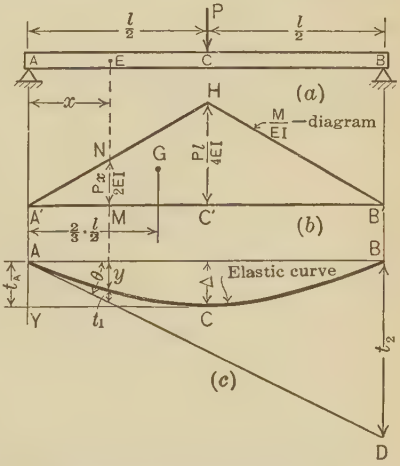

FIG. 272.—Deflection of simple beam, load at center; slope-deviation method.

Let it be required to find the maximum deflection Δ, which occurs at the center of the span. The tangential deviaton of the point A from a tangent at C is t_A (Fig. 272*c*) which is equal in magnitude to Δ, and which may be found from Theorem II of Art. 110. Thus,

$$\Delta = t_A = \text{Moment of area } A'C'H \text{ about } A'$$

$$= \text{Area } A'C'H \text{ times distance to centroid, } G, \text{ of area } A'C'H \text{ from } A'$$

$$= \frac{1}{2}\frac{Pl}{4EI}\frac{l}{2}\cdot\frac{2}{3}\frac{l}{2} = \frac{Pl^3}{48EI}.$$

Next let it be required to find the deflection y of a point E (Fig. 272a) at a distance x from the left support. If a tangent AD is drawn to the elastic curve at A, t_2 is the tangential deviation of B and t_1 is the tangential deviation of the point E. From the geometry of the figure we have,

$$\frac{y + t_1}{t_2} = \frac{x}{l} \quad \text{or} \quad y = t_2 \frac{x}{l} - t_1,$$

and since the slopes of the elastic curve are small we have, $\theta = \frac{t_2}{l}$. Therefore,

$$y = \theta x - t_1.$$

Now θ is equal to the difference of the slopes at A and C and hence, according to Theorem I,

$$\theta = \text{area } A'C'H = \frac{1}{2}\frac{Pl}{4EI}\frac{l}{2} = \frac{1}{16}\frac{Pl^2}{EI},$$

and, according to Theorem II,

$$t_1 = \text{moment of area } A'MN \text{ about } MN$$

$$= \frac{1}{2}\frac{Px}{EI}\cdot\frac{x}{2}\cdot\frac{x}{3} = \frac{Px^3}{12EI}.$$

Therefore,

$$y = \frac{Pl^2x}{16EI} - \frac{Px^3}{12EI},$$

or

$$48EIy = Px(3l^2 - 4x^2),$$

which is the elastic curve equation for the left half of the beam.

112. Cantilever Beam; Concentrated Load at End.—In Fig. 273(a) AB represents the elastic curve of the beam, and in Fig. 273(b), $A'B'H$ represents the $\frac{M}{EI}$-diagram. The beam is assumed to have a constant cross-section and the weight of the beam is neglected. The maximum deflection of the beam may be found as follows.

The tangential deviation, t_A, of A from a tangent drawn at B is equal to the maximum deflection Δ of the beam. Thus from Theorem II we have,

$$\Delta = t_A = \text{moment of area } A'B'H \text{ about } A'$$

$$= \text{area } A'B'H \text{ times distance to centroid}$$

$$= \frac{1}{2}\frac{Pl^2}{EI}\cdot\frac{2}{3}l = \frac{1}{3}\frac{Pl^3}{EI}.$$

The deflection, y, of a point D at the distance x from the free end may be found as follows: The tangential deviation, t_D, of D from a tangent at B is equal to y; thus, using Theorem II we have,

$$y = t_D = \text{moment of area } B'D'EH \text{ about } D'E.$$

Let the area be divided into two triangular areas as indicated by the dotted line. Then

$$y = t_D = \frac{Px}{2EI} \cdot \frac{(l-x)^2}{3} + \frac{Pl}{2EI} \cdot \frac{2}{3}(l-x)^2.$$

Hence,

$$6EIy = P(x^3 - 3l^2x + 2l^3)$$

which is the elastic curve equation for the beam.

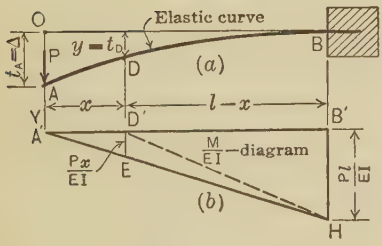

FIG. 273.—Deflection of cantilever beam, load at end; slope-deviation method.

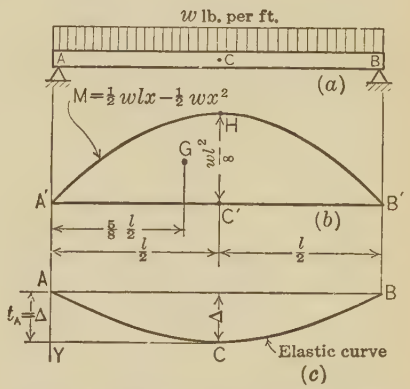

FIG. 274.—Deflection of simple beam; uniform load; slope-deviation method.

113. Simple Beam; Load Distributed Uniformly.—The beam shown in Fig. 274(a) has a constant cross-section and the weight of the beam is negligible. The moment diagram $A'HB'$ (Fig. 274b) is a parabola, the ordinate M to any point being expressed by the equation $M = \frac{1}{2}wlx - \frac{1}{2}wx^2$, and the maximum ordinate being $\frac{1}{8}wl^2$ (see Prob. 152). The elastic curve is represented by the line ACB in Fig. 274(c).

The maximum deflection of the beam may be found as follows: The tangential deviation t_A of the point A (Fig. 274c) from a tangent at C is equal in magnitude to the maximum deflection, Δ, of the beam. Hence according to Theorem II we have

$$\Delta = t_A = \frac{1}{EI} \text{ (moment of area } A'C'H \text{ about } A').$$

Now,

Area $A'C'H = \frac{2}{3}\overline{A'C'} \times \overline{C'H}$ (see Art. 165, Appendix I),

and the distance of the centroid of area $A'C'H$ from A' is $\frac{5}{8}\overline{AC'}$. Therefore,

$$\Delta = t_A = \frac{1}{EI}\left(\frac{2}{3}\cdot\frac{1}{8}wl^2\cdot\frac{l}{2}\cdot\frac{5}{8}\frac{l}{2}\right)$$

$$= \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{384}\frac{Wl^3}{EI}.$$

114. Cantilever Beam; Load Distributed Uniformly.—The elastic curve of the beam is represented by the line AB (Fig. 275a), and the moment-diagram for the beam is shown in Fig. 275(b). The maximum moment, at the wall, is $-\frac{1}{2}wl^2$ and the moment at a distance x from the free end is $-\frac{wx^2}{2}$. Now the tangential deviation of the point A from a tangent drawn at B is equal in magnitude to the maximum deflection Δ. Hence, by integrating equation (226), we have

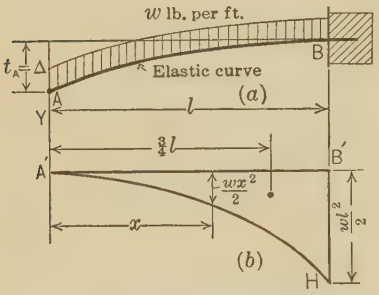

Fig. 275.—Deflection of cantilever beam, uniform load; slope-deviation method.

$$\Delta = t_A = \frac{1}{EI}\int_0^l Mxdx = \frac{1}{EI}\int_0^l \frac{wx^2}{2}xdx$$

$$= \frac{wl^4}{8EI} = \frac{1}{8}\frac{Wl^3}{EI},$$

or, from Theorem II, the value of Δ is found without integrating:

$$\Delta = t_A = \frac{1}{EI}\text{ (moment of area } A'B'H \text{ about } A')$$

$$= \frac{1}{EI}\left(\frac{1}{3}\frac{wl^2}{2}\cdot l\cdot\frac{3}{4}l\right) = \frac{1}{8}\frac{wl^4}{EI}.$$

Conjugate-beam Method

115. Conjugate Beam Defined.—By the conjugate-beam method the deflection at any section of a beam (here called the " given " beam) is found by calculating the bending moment at the corresponding sec-

tion of another beam (here called the substitute or conjugate beam), the conjugate beam being subjected to a distributed load such that the intensity of the load at any section is proportional to the ordinate of the $\frac{M}{EI}$-diagram at that section. That is, the conjugate beam may be assumed to be loaded with sand the depth of which over any section of the beam is, according to a chosen scale, the ordinate of the $\frac{M}{EI}$-diagram at that section. To describe this loading the conjugate beam will be said to be loaded with the $\frac{M}{EI}$-diagram.

By definition the conjugate beam is one that (*a*) has a length, l', which is equal to the length, l, of the given beam, (*b*) is in equilibrium, and (*c*) is so loaded that the bending moment M' at any section is equal to the deflection, y, at the corresponding section in the "given" beam. The defining equations then are: From (*a*)

$$l' = l;$$

from (*b*), since the loads and reactions constitute a parallel system of vectors,[2] there are two equations of equilibrium; namely,

$$\Sigma F' = 0 \quad \text{and} \quad \Sigma M' = 0$$

and from (*c*)

$$M'_x = y_x.$$

The last equation requires, as is shown below, (1) that the vertical shear, V', for any section of the conjugate beam shall be equal to the slope at the corresponding point of the elastic curve of the "given" beam; that is,

$$V' = \frac{dy}{dx} = \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

and (2) that the intensity, w', of the distributed load on the conjugate beam at any section shall be equal to the $\frac{M}{EI}$ for the corresponding section in the given beam; in other words, the conjugate beam is loaded with the $\frac{M}{EI}$-diagram. Thus,

$$w' = \frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

[2] It will be found that the loads and reactions are not forces; that is, they are not expressed in pounds, tons, etc., but for convenience they may be called *elastic forces*, since they involve the elastic properties of the beam.

Proof of Equations (227) *and* (228).—Let a straight beam be subjected to a distributed load only; the intensity of the load at the distance x from the left support is w, and the vertical shear and bending moment for this section are V and M.

Now the difference, dV, in the vertical shears for two sections the distance dx apart is

$$dV = wdx \quad \text{or} \quad w = \frac{dV}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (229)$$

But from Art. 34,

$$V = \frac{dM}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

and hence,

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

The elastic curve equation of the beam is (see Art. 43),

$$\frac{d^2y}{dx^2} = \frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

Thus, a comparison of these last two equations shows that the relation between w and M in eq. (231) is the same as that between $\frac{M}{EI}$ and y in eq. (232). Therefore, if a substitute or conjugate beam is loaded with the $\frac{M}{EI}$ diagram of the actual or "given" beam, the bending moment M' in the conjugate beam would be represented by the deflection y of the given beam.

Further, since the slope θ of the elastic curve at any point is $\theta = \frac{dy}{dx}$, it follows that, if the conjugate beam is loaded so that $M' = y$, then from equation (230)

$$V' = \frac{dy}{dx} = \theta.$$

That is, the vertical shear for any section in the conjugate beam must be equal to the slope of the elastic curve at the corresponding section in the given beam.

It will be noted that, in the conjugate beam method, the actual *elastic* beam is replaced by a *rigid* beam with the elastic properties of the actual beam introduced in the loads and reactions of the rigid beam.

Summarizing: The equations, then, that the *conjugate* beam for any *given* beam must satisfy are

$$l' = l, \quad \Sigma F' = 0, \quad \Sigma M' = 0, \quad M' = y$$

$$V' = \theta, \qquad w' = \frac{M}{EI}.$$

The applications of these equations in finding the deflection of various types of beams is given below. In the following applications the sign of the bending moment is determined as stated in Art. 43; the deflection y is positive in the downward direction, and x is positive to the right, and hence θ is positive when measured in the clockwise direction of rotation.

APPLICATIONS

116. Simple Beam; Concentrated Load at Mid-span.—The beam is bent as shown in Fig. 276(a). The bending moment diagram (M-diagram) is shown in Fig. 276(b) and since the beam has a constant cross-section the $\frac{M}{EI}$-diagram will have the same form as the M-diagram. The conjugate beam is shown in Fig. 276(c); the bending moment at the mid-span section of the conjugate beam is equal to the deflection of the mid-span section (maximum deflection) of the given beam, provided that the conjugate beam is made to satisfy equations of Art. 115. This may be done by selecting the beam as shown in Fig. 276(c) in which $l' = l$; the distributed load is the $\frac{M}{EI}$-diagram; since $M' = y$, and since $y = 0$ at the ends of the "given" beam, the moment M' at the ends of the conjugate beam must also equal zero, and hence the ends of the conjugate beam are subjected to zero moments, that is, the ends are free to turn as shown in Fig. 276(c); further, since $V' = \theta$ and since θ is not zero at the ends of the given beam, there must be vertical shears at the ends of the conjugate beam, and these could be produced by reactions R'_1 and R'_2 of supports at the ends. Thus, if the

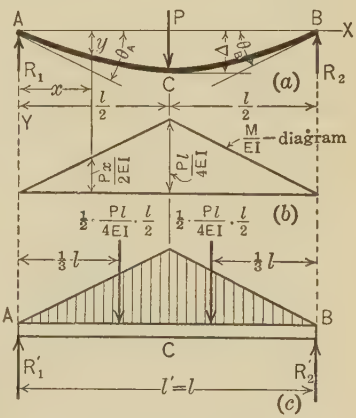

FIG. 276.—Deflection of simple beam, load at center; conjugate-beam method.

"given" beam is a simple beam the conjugate beam also is a simple beam.

Now the maximum deflection Δ of the given beam occurs at the midspan section and hence it is equal to the moment, M'_c, at the center of the conjugate beam, which is the moment of the couple shown in Fig. 276(*c*). Thus,

$$\Delta = M'_c = \frac{1}{16}\frac{Pl^2}{EI}\cdot\frac{2}{3}\frac{l}{2} = \frac{1}{48}\frac{Pl^3}{EI}.$$

Further, since the conjugate beam is in equilibrium the values of R'_1 and R'_2 may be found from the equations $\Sigma F' = 0$ and $\Sigma M' = 0$. Thus, from $\Sigma M' = 0$, or from the conditions of symmetry, we obtain, $R'_1 = R'_2$. And from $\Sigma F' = 0$ we obtain the values of R'_1 and R'_2, which are equal to the vertical shears at the ends of the conjugate beam and hence equal in magnitude to the slopes, θ_A and θ_B, at the ends of the "given" beam. Thus,

$$\Sigma F' = 2R'_1 - \frac{1}{2}\frac{Pl}{4EI}\cdot l = 0,$$

whence,

$$R'_1 = \frac{1}{16}\frac{Pl^2}{EI},$$

as is evident from inspection of Fig. 276(*c*). Therefore,

$$\theta_A = \frac{1}{16}\frac{Pl^2}{EI}.$$

Deflection at Any Point.—The deflection y of the "given" beam at a distance x from the left support is equal to the bending moment M'_x of the conjugate beam which, as indicated in Fig. 277, is

$$y = M'_x = R'_1 x - \frac{1}{2}\frac{Px^2}{2}\cdot\frac{1}{3}x$$

$$= \frac{Pl^2x}{16EI} - \frac{1}{12}\frac{Px^3}{EI} = \frac{P}{4EI}\left(\frac{l^2x}{4} - \frac{x^3}{3}\right).$$

This is the equation of the elastic curve of the left half of the given beam.

117. Cantilever Beam; Concentrated Load at End.—The given beam is assumed to have a constant cross-section; it bends as shown in Fig. 278(*a*). The conjugate beam is shown in Fig. 278(*b*); the distributed load on the conjugate beam is the $\frac{M}{EI}$-diagram and is an upward load

since M is negative. The end conditions of the conjugate beam are found by making the beam satisfy the fundamental equations of Art. 115. Thus, since $M' = y$ and $V' = \theta$, and, further, since y and θ are not zero at the free end of the "given" beam, then the moment M'_0 and the shear V'_0 at the left end of the conjugate beam are not zero. M'_0 and V'_0 may be produced by fixing the beam at the left end. On the other hand, since the θ and y at the right end of the "given" beam are zero, the M' and V' at the right end of the conjugate beam must be zero, and this condition will exist if the right end is free. Therefore

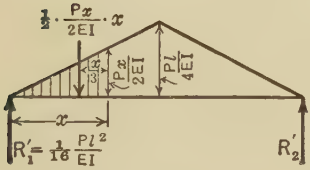

FIG. 277.

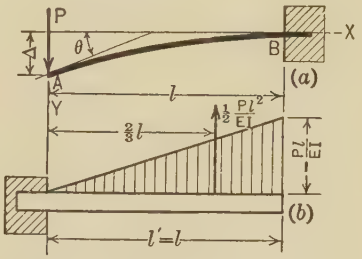

FIG. 278.—Deflection of cantilever beam, load at end; conjugate-beam method.

the conjugate beam corresponding to a cantilever beam is another cantilever beam with the end conditions interchanged.

Now the maximum deflection, Δ, of the "given" beam is equal to the moment M'_0 at the left end of the conjugate beam. Thus,

$$\Delta = M'_0 = \frac{1}{2}\frac{Pl^2}{EI}\cdot\frac{2}{3}l = \frac{1}{3}\frac{Pl^3}{EI}.$$

The slope, θ, at the free end of the given beam may be found, if desired, by calculating the shear V'_0 at the left end of the conjugate beam; the value found is

$$\theta = V'_0 = \frac{1}{2}\frac{Pl^2}{EI}.$$

The elastic curve equation may be found in a manner similar to that used in the preceding article.

118. Simple Beam; Load Distributed Uniformly.—The beam is assumed to have a constant cross-section; it deflects as shown in Fig. 279(a). The corresponding conjugate beam is shown in Fig. 279(b), the distributed load being the $\frac{M}{EI}$-diagram, which is a parabolic area,

the maximum ordinate of the diagram being $\frac{1}{8}\frac{wl^2}{EI}$ (see Prob. 152).

The maximum deflection Δ is equal to the moment at the center of the conjugate beam, which is the moment of the couple shown in Fig. 279(*b*). Thus,

$$\Delta = M'c = \frac{2}{3}\cdot\frac{1}{8}\frac{wl^2}{EI}\cdot\frac{l}{2}\cdot\frac{5}{8}\frac{l}{2} = \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{384}\frac{Wl^3}{EI}.$$

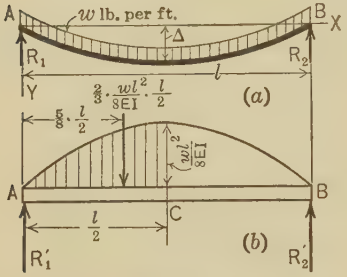

FIG. 279.—Deflection of simple beam, uniform load; conjugate-beam method.

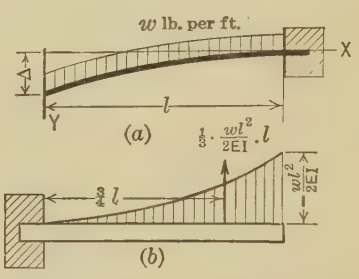

FIG. 280.—Deflection of cantilever beam, uniform load; conjugate-beam method.

119. Cantilever Beam; Load Distributed Uniformly.—The beam has a constant cross-section; it deflects as shown in Fig. 280(*a*). The corresponding conjugate beam is shown in Fig. 280(*b*). The maximum deflection Δ is equal to the moment, M'_0, at the left end of the conjugate beam. Thus,

$$\Delta = M'_0 = \frac{1}{3}\cdot\frac{wl^3}{2EI}\cdot\frac{3}{4}\,l = \frac{1}{8}\frac{wl^4}{EI} = \frac{1}{8}\frac{Wl^3}{EI}.$$

120. Simple Beam; Concentrated Load at Any Point.—Let the load act at a distance a from the left end and b from the right end; the deflected beam is shown in Fig. 281(*a*). The $\frac{M}{EI}$-diagram is shown in Fig. 281(*b*) as a distributed load acting on the conjugate beam; the "given" beam is assumed to have a constant cross-section.

The total load is $\frac{1}{2}\frac{Pab}{EIl}\cdot l$ and its action line is at a distance of $\frac{1}{3}(l + a)$ from the left end. From the equations of equilibrium, $\Sigma F' = 0$ and $\Sigma M' = 0$, we find the values of R'_1 and R'_2 (and hence of θ_A and θ_B) to be,

$$R'_1 = \frac{1}{3}\frac{l+b}{l}\frac{Pab}{2EI} \quad \text{and} \quad R'_2 = \frac{1}{3}\frac{l+a}{l}\frac{Pab}{2EI}.$$

The deflection y at the distance x from the left end of the given beam is equal to the bending moment, M'_x, in the conjugate beam. Thus,

$$y = M'_x = R'_1 x - \frac{1}{2}\frac{Pbx^2}{EIl}\frac{1}{3}x$$

$$= \frac{1}{3}\frac{l+b}{l}\frac{Pab}{2EI}x - \frac{1}{6}\frac{Pbx^3}{lEI}$$

$$= \frac{Pbx}{6EIl}[a(l+b) - x^2], \quad . \ . \ . \ . \quad (233)$$

which is the elastic curve equation of that portion of the beam to the left of the load P.

The maximum deflection may be found as follows: The maximum moment in the conjugate beam is equal to the maximum deflection in the given beam; now the section of maximum moment in the conjugate beam is the section for which the vertical shear in the conjugate beam is zero. Thus, the value of x' in the following equation locates the point of maximum deflection in the given beam.

$$V'_x = R'_1 - \frac{Pbx^2}{2lEI} = 0$$

$$= \frac{1}{3}\frac{l+b}{l}\frac{Pab}{2EI} - \frac{Pbx^2}{2lEI} = 0,$$

whence,

$$x = \sqrt{\tfrac{1}{3}a(l+b)}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (234)$$

If this value of x is substituted in equation (233), the resulting value of y is the maximum deflection Δ (see Art. 48 for the value of Δ).

PROBLEM

430. Find, from equations (233) and (234), the maximum deflection of a simple beam subjected to a concentrated load at the center of the span, and compare the result with that given in Art. 45.

121. Simple Beam; Cross-section Not Constant.—Let a concentrated load act at the center of a simple beam, and let the moment of inertia for any section in the center half of the beam be I and for any section in the outer quarters $\frac{1}{2}I$ as indicated in Fig. 282(a). The moment diagram is shown in Fig. 282(b), and the $\frac{M}{EI}$-diagram is shown

in Fig. 282(*c*) acting as a load on the conjugate beam. The $\frac{M}{EI}$-diagram may be divided into four triangles, I, II, III and IV, as shown in Fig. 282(*c*). From the equilibrium equation $\Sigma F' = 0$, we have

$$R'_1 = Q_1 + Q_2 = \frac{5}{64}\frac{Pl^2}{EI}.$$

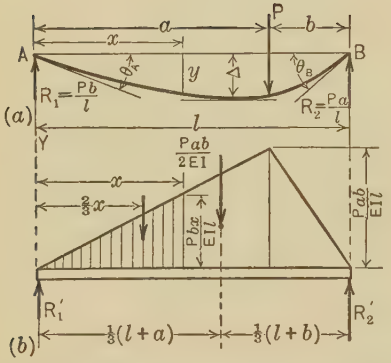

FIG. 281.—Deflection of simple beam, load at any point; conjugate-beam method.

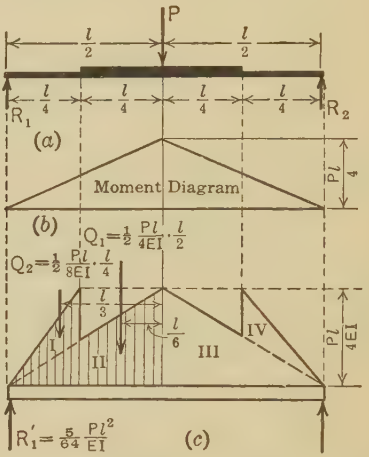

FIG. 282.—Deflection of beam with variable section; conjugate-beam method.

and the maximum deflection Δ, which is equal to the bending moment M'_c at the center of the conjugate beam is,

$$\Delta = M'_c = R'_1\frac{l}{2} - Q_1\frac{1}{6}l - Q_2\frac{1}{3}l$$

$$= \left(\frac{5}{128} - \frac{1}{96} - \frac{1}{192}\right)\frac{Pl^3}{EI} = \frac{3}{128}\frac{Pl^3}{EI}.$$

PROBLEMS

431. A simple beam is subjected to a concentrated load P at the center of the span; the moment of inertia of the left half of the beam is I and of the right half is $\frac{1}{2}I$. Derive the expression for the deflection of the center of the beam.

Ans. $\frac{1}{32}\frac{Pl^3}{EI}$.

NOTE.—The problems in Chapter V may also be used in connection with this chapter.

CHAPTER XIII

STATICALLY INDETERMINATE BEAMS

(Moment-area Method)

122. Introduction.—As explained in Art. 51, a statically indeterminate beam is one for which the number of reactions is greater than the number of equations of equilibrium; in order to determine the reactions for such a beam the equation of the elastic curve of the beam is needed, in addition to the equations of equilibrium. The use, by the double integration method, of the general equation of the elastic curve for this purpose, is discussed in Chapter VI. In this chapter the moment-area method will be used to furnish sufficient equations, in addition to the equations of equilibrium, to determine the reactions of statically indeterminate beams. Further, two methods of using moment-areas will be employed; namely, the slope-deviation method and the conjugate-beam method; the equations and theorems used in the slope-deviation method are stated in Art. 110, and the equations used in the conjugate-beam method are given in Art. 115.

Slope-deviation Method

123. Beam Fixed at Both Ends; Load Concentrated at Mid-span.—The beam is shown in its deflected form in Fig. 283(*a*). The beam may be considered to have been a simple beam to which negative end-moments have been applied which cause zero slopes at the ends as indicated in Fig. 283(*b*). The moment-diagram may be found by superimposing the positive moment-diagram for a simple beam subjected to a concentrated load at the mid-span, the maximum ordinate to which is at the mid-span and equal to $\frac{Pl}{4}$, and a negative moment-diagram consisting of a rectangle, the constant ordinate to which is the unknown end-moment M_0; this superimposed diagram is shown in Fig. 283(*c*), the end-moments being known to be equal from the conditions of symmetry (or of equilibrium). Since the beam has a

constant cross-section, Fig. 283(*c*) also represents the $\frac{M}{EI}$-diagram. Since the beam is fixed at both ends, we have

$$\text{Change of slope from } A \text{ to } B = 0,$$

and hence, from Theorem I of Art. 110, we have

$$\text{Total area of } \frac{M}{EI}\text{-diagram} = 0,$$

that is,

$$\frac{1}{EI}\left(\frac{1}{2}\frac{Pl}{4}\cdot l + M_0 l\right) = 0,$$

and hence,

$$M_0 = -\tfrac{1}{8}Pl.$$

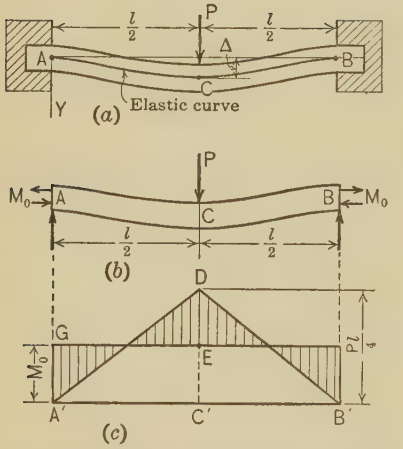

FIG. 283.—Moments at ends of fixed beam; slope-deviation method.

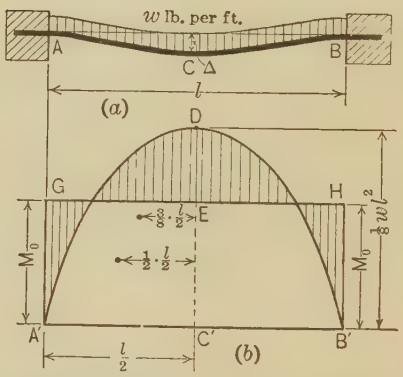

FIG. 284.—Moments at ends of fixed beam; slope-deviation method.

According to Theorem II of Art. 110 the maximum deflection, Δ, which occurs at C (Fig. 283*a*), is

$$\Delta = \text{moment of } \frac{M}{EI}\text{-diagram from } A' \text{ to } C' \text{ about } C'$$

$$= \frac{1}{EI}\,(\text{moment of area } A'DC' \text{ about } C'D + \text{moment of area } A'GEC' \text{ about } C'E)$$

$$= \frac{1}{EI}\left(-\frac{1}{2}\frac{Pl}{4}\cdot\frac{l}{2}\cdot\frac{1}{3}\frac{l}{2} + \frac{Pl}{8}\cdot\frac{l}{2}\cdot\frac{l}{4}\right) = \frac{Pl^3}{192EI}.$$

124. Beam Fixed at Both Ends; Load Uniformly Distributed.—The deflected beam is shown in Fig. 284(*a*). The $\frac{M}{EI}$-diagram is shown in Fig. 284(*b*); it is considered to be composed of two parts, as in the preceding article, consisting of the positive area $A'DB'$ and the negative area $A'GHB'$.

Since the change in slope from A to B (Fig. 284*a*) is equal to zero we have, from Theorem I of Art. 110,

$$\text{Total area of } \frac{M}{EI}\text{-diagram} = 0,$$

that is,

$$\frac{2}{3}\cdot\frac{wl^2}{8EI}\cdot l + \frac{M_0 l}{EI} = 0.$$

Hence,

$$M_0 = -\frac{wl^2}{12}.$$

And from Theorem II of Art. 110, the maximum deflection, Δ, is

$$\Delta = \text{moment of area } A'DC' \text{ about } C'D - \text{moment of area } A'GEC' \text{ about } C'E$$

$$= -\frac{2}{3}\frac{wl^2}{8EI}\cdot\frac{l}{2}\cdot\frac{3}{16}l + \frac{l}{2}\left(\frac{wl^2}{12EI}\right)\cdot\frac{l}{4} = \frac{wl^4}{384EI}.$$

125. Beam Fixed at One End, Supported at Other End; Load Concentrated at Mid-span.—The deflected beam is shown in Fig. 285(*a*), and the $\frac{M}{EI}$-diagram considered to be made up of two parts, as discussed in Art. 123, is shown in Fig. 285(*b*). Thus the moment-diagram consists of a triangle with the maximum ordinate at the center of the span, the same as that for a simple beam with a concentrated load P at the mid-span, and a negative triangular diagram the ordinates to which vary from M_0 at the wall to zero at the left end. Further, since the cross-section of the beam is constant, the $\frac{M}{EI}$-diagram has the same form as the moment-diagram.

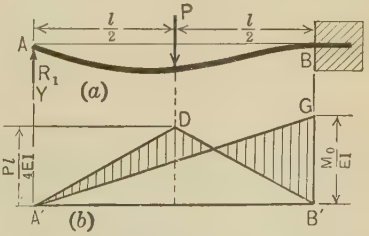

FIG. 285.—Moment at fixed end of beam; slope-deviation method.

Since the tangential deviation, t_A, of the point A from a tangent drawn at B is equal to zero, we have, from Theorem II of Art. 110,

$$t_A = \text{moment of } \frac{M}{EI}\text{-diagram about } A' = 0$$

$$= (\text{moment of area } A'DB' \text{ about } A' + \text{moment of area } A'GB' \text{ about } A') = 0$$

$$= \frac{1}{2}\frac{Pl}{4EI}\cdot l\cdot\frac{l}{2} + \frac{1}{2}\cdot\frac{M_0 l}{EI}\cdot\frac{2}{3}l = 0.$$

Hence,

$$M_0 = -\tfrac{3}{16}Pl.$$

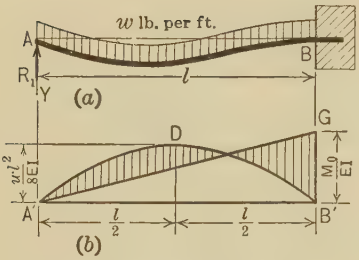

FIG. 286.—Moment at fixed end of beam; slope-deviation method.

126. Beam Fixed at One End, Supported at Other End; Load Uniformly Distributed.—The deflected beam is shown in Fig. 286(a) and the $\frac{M}{EI}$-diagram is shown in Fig. 286(b). The tangential deviation of the point A from a tangent at B is equal to zero. Hence, from Theorem II of Art. 110 we have

$$t_A = \text{moment of } \frac{M}{EI}\text{-diagram about } A' = 0$$

$$= \text{moment of area } A'DB' \text{ about } A' + \text{moment of area } A'GB' \text{ about } A' = 0$$

$$= \frac{2}{3}\frac{wl^2}{8EI}\cdot l\cdot\frac{l}{2} + \frac{1}{2}\frac{M_0}{EI}\cdot l\cdot\frac{2}{3}l = 0.$$

Hence,

$$M_0 = -\frac{wl^2}{8}.$$

127. Continuous Beam; Theorem of Three Moments.—Let Fig. 287(a) represent a continuous beam subjected to uniformly distributed loads; the beam has a constant cross-section and rests on supports that are on the same level.

The elastic curve of two adjacent spans of the beam is shown in Fig. 287(b). If, over the supports, hinges are introduced, as indicated in Fig. 287(c) and then external end-moments, M_1, M_2 and M_3, are

applied as shown, the beam will act in all ways as the original beam acts, provided that the values of M_1, M_2 and M_3 are the values of the bending moments at the supports of the original beam. The moment diagram formed by superimposing the positive and negative moments, caused by the loading shown in Fig. 287(*c*) is represented in Fig. 287(*d*).

Let a tangent be drawn to the elastic curve at the support B (Fig. 287*b*); since the elastic curve is continuous over the support, the tangent

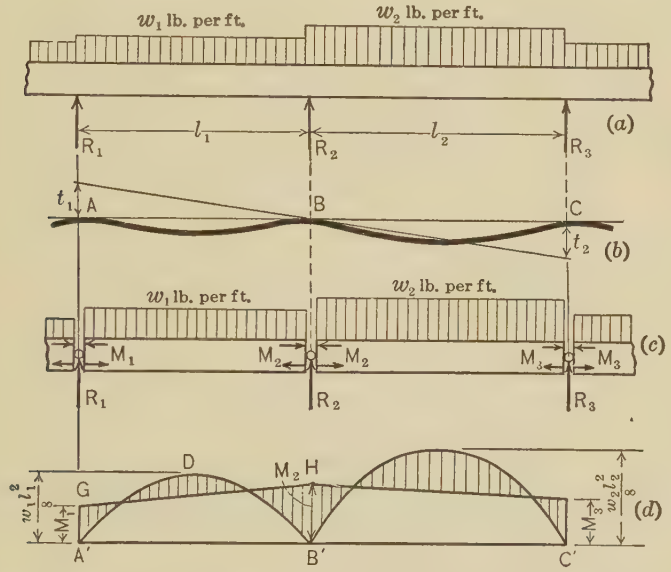

FIG. 287.—Moments at supports of continuous beam; slope-deviation method.

to the elastic curve at B is common to the two curves of the adjacent spans l_1 and l_2.

Now, from theorem II of Art 110, the tangential deviation t_1 of the point A from a tangent at B is

$$t_1 = \frac{1}{EI} \begin{pmatrix} \text{moment, about } A', \text{ of area } A'DB' + \text{moment} \\ \text{about } A', \text{ of area } A'GHB' \end{pmatrix}$$

$$= \frac{1}{EI}\left(\frac{2}{3}\frac{w_1 l_1^2}{8}\cdot l_1 \cdot \frac{l_1}{2} + \frac{1}{2}M_1 l_1 \frac{l_1}{3} + \frac{1}{2}M_2 l_1 \cdot \frac{2l_1}{3}\right)$$

$$= \frac{1}{EI}\left(\frac{w_1 l_1^4}{24} + \frac{M_1 l_1^2}{6} + \frac{M_2 l_1^2}{3}\right).$$

Similarly, taking moments about C' of areas of the moment diagram for the next span, we have

$$t_2 = \frac{1}{EI}\left(\frac{2}{3}\frac{w_2 l_2^2}{8}\cdot l_2\cdot\frac{l_2}{2} + \frac{1}{2}M_3 l_2\cdot\frac{l_2}{3} + \frac{1}{2}M_2\cdot l_2\frac{2l_2}{3}\right)$$

$$= \frac{1}{EI}\left(\frac{w_2 l_2^4}{24} + \frac{M_2 l_2^2}{3} + \frac{M_3 l_2^2}{6}\right).$$

Now, from similar triangles,

$$\frac{t_1}{l_1} = \frac{-t_2}{l_2},$$

whence,

$$\frac{w_1 l_1^3}{24} + \frac{M_1 l_1}{6} + \frac{M_2 l_1}{3} = -\left(\frac{w_2 l_2^3}{24} + \frac{M_3 l_2}{6} + \frac{M_2 l_2}{3}\right),$$

or,

$$\frac{M_1 l_1}{6} + \frac{M_2(l_1 + l_2)}{3} + \frac{M_3 l_2}{6} = -\frac{w_1 l_1^3}{24} - \frac{w_2 l_2^3}{24}.$$

Therefore,

$$M_1 l_1 + 2M_2(l_1 + l_2) + \boldsymbol{M_3 l_2} = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4}.$$

This equation expresses the theorem of three moments for a continuous beam subjected to uniform loads, the supports being on the same level. For use of the theorem see Art. 58.

CONJUGATE-BEAM METHOD

128. Beam Fixed at Both Ends; Load Distributed Uniformly.—The deflected beam is shown in Fig. 288(a). In Fig. 288(b) the beam is shown acted on by an equivalent force system which converts the beam into a simple beam subjected to end-moments M_0. The moment-diagram for the beam in Fig. 288(b), and hence also for the original beam, is shown in Fig. 288(c); the positive moment diagram of the vertical forces is a parabola the same as that for a simple beam, and the negative moment diagram of the end-moments is a rectangle. Further, since the beam has a constant cross-section, the $\frac{M}{EI}$-diagram has the same form as the moment-diagram.

Now as discussed in Art. 117, the conjugate beam for a fixed-ended "given" beam is a beam with free ends, that is, the $\frac{M}{EI}$-diagram,

acting alone as a distributed load on the conjugate beam, must hold the conjugate beam in equilibrium as indicated in Fig. 288(*d*), and hence the $\frac{M}{EI}$-load diagram must be so adjusted that the sum of all the loads is equal to zero ($\Sigma F' = 0$, Art 115); that is, in Fig. 288(*c*), the $\frac{M}{EI}$-load of the parabolic diagram must be equal and opposite to the $\frac{M}{EI}$-load of the rectangular diagram. Hence

$$\frac{2}{3}\frac{wl^2}{8EI} \cdot l = -\frac{M_0}{EI} l,$$

whence,

$$M_0 = -\tfrac{1}{12} wl^2.$$

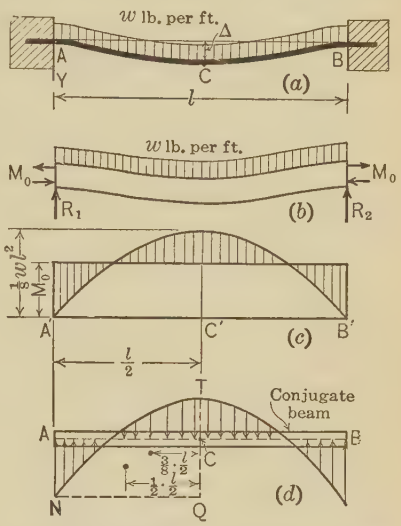

FIG. 288.—Moments at ends of fixed beam; conjugate-beam method.

Maximum Deflection. — The maximum deflection Δ of the given beam is equal to the bending moment M'_c at the center of the conjugate beam. In finding M'_c the load to the left of the center may be considered to consist of two parts: the rectangle $ACQN$ (Fig. 288*d*) acting as an upward load, and the parabolic area NTQ acting as a downward load; these two areas are equal since the beam is in equilibrium. Taking moments about C' we have,

$$\Delta = M'_c = \frac{1}{12}\frac{wl^2}{EI}\cdot\frac{l}{2}\cdot\frac{l}{4} - \frac{2}{3}\cdot\frac{1}{8}\frac{wl^2}{EI}\cdot\frac{l}{2}\cdot\frac{3}{8}\frac{l}{2}$$

$$= \frac{wl^4}{384EI} = \frac{Wl^3}{384EI}.$$

129. Beam Fixed at Both Ends; Load Concentrated at Mid-span.— The deflected beam is shown in Fig. 289(*a*). A force system that would produce the same deflections as occur in the "given" beam is shown in Fig. 289(*b*), and the superimposed moment-diagram is shown in Fig. 289(*c*). The conjugate beam (Fig. 289*d*) for a fixed-ended "given" beam has free ends (see Art. 117), and hence the $\frac{M}{EI}$-diagram

must hold the conjugate beam in equilibrium; that is, the total $\frac{M}{EI}$-load must equal zero ($\Sigma F' = 0$, Art. 115). Hence,

$$\frac{M_0}{EI} l + \frac{1}{2} \cdot \frac{1}{4} \frac{Pl}{EI} \cdot l = 0,$$

whence,

$$M_0 = -\tfrac{1}{8} Pl.$$

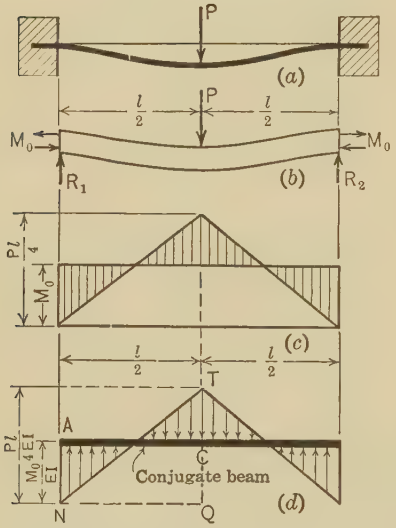

FIG. 289.—Moments at ends of fixed beam; conjugate-beam method.

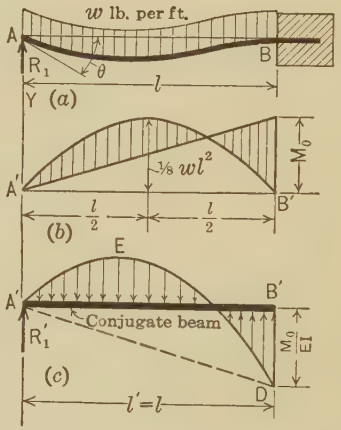

FIG. 290.—Moment at fixed end of beam; conjugate-beam method.

The maximum deflection Δ is equal to the bending moment M'_c at the center of the conjugate beam. Thus,

$$\Delta = M'_c = (\text{moment } ACQN - \text{moment } NTQ)$$

$$= \frac{1}{8} \frac{Pl}{EI} \frac{l}{2} \cdot \frac{l}{4} - \frac{1}{2} \frac{1}{4} \frac{Pl}{EI} \cdot \frac{l}{2} \cdot \frac{1}{3} \frac{l}{2} = \frac{1}{192} \frac{Pl^3}{EI}.$$

130. Beam Fixed at One End; Supported at Other End; Load Uniformly Distributed.—The deflected beam is shown in Fig. 290(*a*). The superimposed moment-diagram is shown in Fig. 290(*b*).

Since the deflection and slope at the right end of the "given" beam are zero, the moment and shear at the right end of the conjugate beam must be zero, and hence the end is free (see Art. 117); whereas,

the left end of the conjugate beam has zero moment since the deflection at A is zero ($M' = y = 0$) but the shear at the left end is not zero ($V' = \theta$), and hence the conjugate beam is supported at the left end. Further, it is subjected to the distributed $\frac{M}{EI}$-load as indicated in Fig. 290(c).

The value of M_0 may be found by applying one of the equations of equilibrium ($\Sigma M' = 0$). Thus if moments are taken about the left end of the conjugate beam the moment of the triangular area $A'B'D$ plus the moment of the parabolic area $A'ED$ must equal zero. Hence,

$$\frac{1}{2}\frac{M_0}{EI}l\cdot\frac{2}{3}l + \frac{2}{3}\cdot\frac{1}{8}\frac{wl^2}{EI}l\cdot\frac{l}{2} = 0,$$

whence,

$$M_0 = -\frac{1}{8}wl^2.$$

Now R'_1 (and hence θ) may be found, if desired, from the other equilibrium equation ($\Sigma F' = 0$).

131. Continuous Beam; Theorem of Three Moments.—Fig. 291(a) shows two spans of a continuous beam subjected to distributed loads; the supports are on the same level, and the beam has a constant cross-section. If, over the supports, hinges are introduced and then end-moments, M_1, M_2 and M_3, are applied, as in Fig. 291(c), the beam will act in all ways as the original beam acts, provided that the value of M_1, M_2 and M_3 are the values of the bending moments at the supports of the original beam.

The superimposed moment diagram for the beam in Fig. 291(c) (and hence also of the original beam) is shown in Fig. 291(d), and the $\frac{M}{EI}$-diagram will have the same form as that of the moment-diagram. The $\frac{M}{EI}$-diagram is the distributed load on the conjugate beam (Fig. 291e); now the conjugate beam has hinges and no supports at the points corresponding to the sections over the support of the original beam. This may be explained as follows: The deflection at the supports of the "given" beam, and therefore the moments at the supports in the conjugate beam, are zero; the slope of the given beam at a support may be different from zero (as indicated in Fig. 291b) but must have the same value immediately to the left and to the right of the point since the beam is continuous over the support, and therefore the shear in the conjugate beam may have a value other than zero but must have

the same value immediately to the left and to the right of the point. An unsupported hinge is the simplest arrangement by which this condition can be established.

By expressing the fact that the $\frac{M}{EI}$-load must be so adjusted that the reaction of the support (B say, Fig. 291e) is zero, the theorem of

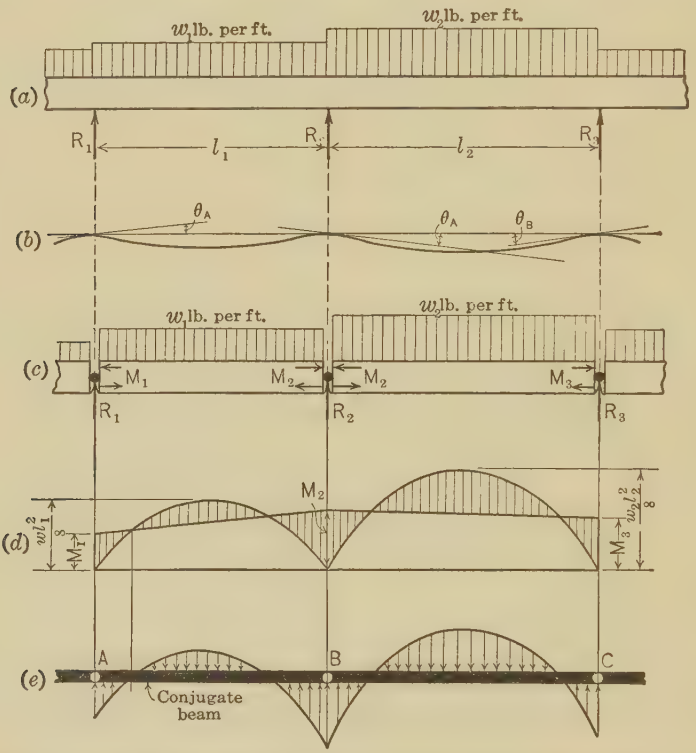

FIG. 291.—Moments at supports of continuous beam; conjugate-beam method.

three moments is found as follows: Let it be assumed first that all the $\frac{M}{EI}$-loads are acting downward, then since there is no moment at B the reaction at B would be such that its moment about A would be equal to the moment, about A, of the $\frac{M}{EI}$-loads on span l_1, and its moment about C would equal the moment, about C, of the $\frac{M}{EI}$-loads on

l_2. Therefore, writing the expression for the reaction and equating it to zero, we have

$$\frac{1}{l_1}\left(\frac{2}{3}\cdot\frac{w_1 l_1^2}{8}\cdot l_1\cdot\frac{l_1}{2}+\frac{1}{2}M_2 l_1\cdot\frac{2}{3}l_1+\frac{1}{2}M_1 l_1\cdot\frac{1}{3}l_1\right)$$

$$+\frac{1}{l_2}\left(\frac{2}{3}\frac{w_2 l_2^2}{8}\cdot l_2\cdot\frac{l_2}{2}+\frac{1}{2}M_2 l_2\cdot\frac{2}{3}l_2+\frac{1}{2}M_3 l_2\frac{1}{3}l_2\right)=0,$$

whence

$$M_1 l_1+2M_2(l_1+l_2)+M_3 l_2=-\tfrac{1}{4}w_1 l_1^3-\tfrac{1}{4}w_2 l_2^3,$$

which expresses the theorem of three moments. For use of the theorem see Art. 58.

NOTE.—The problems in Chapter VI may also be used in connection with this chapter.

CHAPTER XIV

LIMITATIONS OF THE FLEXURE FORMULA FOR SECTIONS HAVING ONLY ONE AXIS OF SYMMETRY [1]

132. Bending Axis and Shear Center Defined.—The *bending axis* of a beam is the longitudinal line in the beam through which the transverse bending loads must pass in order to prevent the beam from twisting as it bends. The *shear center or center of twist* for any transverse cross-section of the beam is the point in the section through which the bending axis passes.

133. The Shear Center and the Flexure Formula.—In the derivation of the flexure formula, $s = \frac{Mc}{I}$, it was assumed that the centroid of the cross-section of the beam was the shear center and that the longitudinal centroidal axis of a prismatic beam was the bending axis. It was further assumed that the neutral axis was perpendicular to the plane of the loads. These assumptions are correct provided that the cross-section has two axes of symmetry, and that the plane of loads contains one of these axes of symmetry (or principal [2] axes) as will be shown in the next chapter.

Further, if the cross-section of the beam has only one axis of symmetry the beam will bend without twisting and the flexure formula will be applicable provided that the plane of the loads contains this axis of

[1] In reading this and the following chapters the student will do well to study again Art. 83 of Chap. IX and to observe how the method of attack there explained is applied in the solution of the problems of determining the stresses in the members considered in this and the remaining chapters. For a somewhat more detailed treatment of the topics in the following chapters and for references for further study see the author's "Advanced Mechanics of Materials," John Wiley & Sons, New York.

[2] A principal axis of inertia of an area with respect to any point in the area is an axis which passes through the point and about which the moment of inertia is either greater than or less than that about any other axis passing through the point. An axis of symmetry is always a principal axis; however, an area always has two principal axes with respect to any point in the area whether or not the area has axes of symmetry. Further, the two principal axes are always perpendicular to each other, and the moment of inertia about one is a maximum and about the other a minimum.

symmetry as illustrated in (*a*) and (*b*) of Fig. 292. If it is desired, however, to use the beam so that the axis of symmetry is the neutral axis, the plane of the loads must be perpendicular to the neutral axis but must *not* pass through the longitudinal centroidal axis of the beam if twisting is to be avoided. In other words the bending axis and longitudinal centroidal axis are not coincident, as is indicated in Fig. 292.

The location of the shear center for certain sections, and its significance in the application of the flexure formula, are discussed briefly in the following articles.

134. Approximate Location of the Shear Center for a Channel Section.—The channel section is a common section having one axis of symmetry; it is designed to be used with the axis of symmetry as the

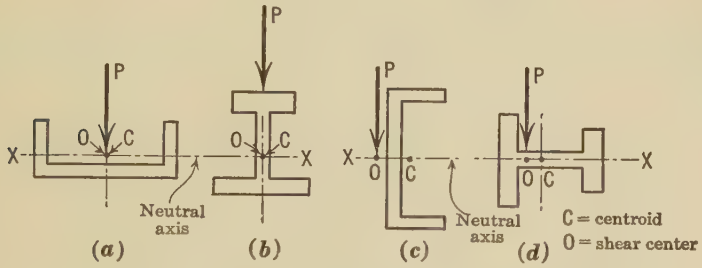

FIG. 292.—Sections having only one axis of symmetry.

neutral axis. A simple method of determining the approximate position of its shear center is as follows:

Let Fig. 293(*a*) represent a channel beam (assumed to be a cantilever beam for convenience) subjected to loads P_1, P_2, etc., that lie in a plane, parallel to the web, so located that the loads cause the channel to bend without twisting, and cause stresses on any transverse section in accordance with the flexure formula as indicated by the stresses shown acting on section AB (Fig. 293*b*). At any transverse section, such as CD, the bending stresses and shearing stresses together hold in equilibrium the external forces P_1, P_2, etc., that lie to one side of the section. The vertical shear or resultant of the loads P_1, P_2, etc., is denoted by V. By introducing two equal and opposite forces at the shear center O, each equal to V, the vertical shear may be resolved into a force $V_1 = V$ at O and a bending couple whose moment is M; the forces of the couple are V and V_2. This bending moment M is held in equilibrium by the bending stresses on the section CD similar to those shown on section AB (these bending stresses on CD are not shown in Fig. 293(*b*)). The vertical force $V_1 = V$ causes an equal (but opposite) resisting shear V_r,

which will be assumed to be developed only in the web of the section and to act along the center line of the web, since the web is narrow. The forces V_1 and V_r form a twisting couple whose moment is Ve, where e is the distance from the center line of the web to the shear center O. For equilibrium this twisting moment must be equal (and opposite) to the twisting moment of the lateral shearing forces, H, in the flanges of the channel. That is,

$$Ve = Hh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (235)$$

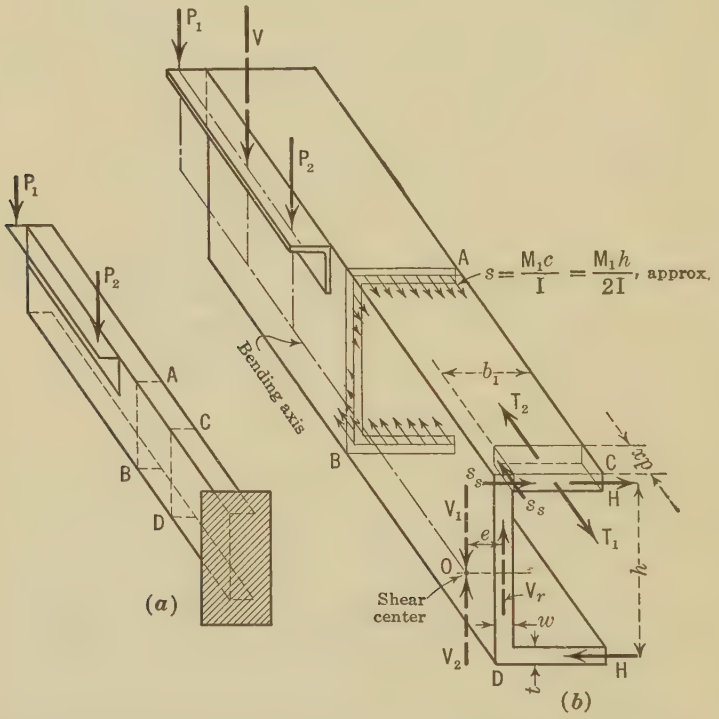

FIG. 293.—Bending axis and shear center for a channel beam.

where h is the distance between the center lines of the flanges, but may be considered without serious error to be the full depth of the section.

The fact that the lateral shearing forces H exist in the flanges may be shown, and a value for H may be found, as follows: The difference $T_1 - T_2 = dT$ of the total bending stress in two sections of a flange the distance dx apart causes a longitudinal shearing stress on the area connecting the flange with the web; this area is tdx, where t is the thickness of the flange (in a rolled-steel channel t will be taken as the mean

thickness of flange). If s_s denotes the shearing unit-stress on this area, then for equilibrium,

$$s_s t dx = dT. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (236)$$

But the total bending stress T in one flange at any section is $b_1 t s$, where b_1 is the width of the outstanding flange, and $s = \frac{M\frac{h}{2}}{I}$ is the bending unit-stress in the flange; I is the moment of inertia of the whole cross-section of the channel with respect to the centroidal axis parallel to the flanges. Thus

$$T = \frac{Mh}{2I} b_1 t \quad \text{and} \quad dT = \frac{dMhb_1 t}{2I},$$

where dM is the difference in the bending moments at the two sections. Therefore

$$s_s t dx = \frac{dMhb_1 t}{2I},$$

or

$$s_s = \frac{dM}{dx}\frac{hb_1}{2I} = \frac{Vhb_1}{2I},$$

in which the familiar relation $\frac{dM}{dx} = V$ is used. As noted above, s_s is the value of the longitudinal shearing unit-stress on the area connecting the web and flange. But an equal shearing unit-stress must exist at the same point on the transverse section of the flange; this lateral shearing unit-stress decreases to zero at the outer edge of the flange, and the assumption will be made that it decreases directly as the distance from the web. Hence the average shearing unit-stress in each flange is $\frac{Vhb_1}{4I}$. The total lateral shear H, in each flange then is

$$H = \frac{Vhb_1}{4I} b_1 t$$

and the moment of these H forces is

$$Hh = \frac{Vb_1^2 h^2 t}{4I}.$$

Since the moment Ve must balance this moment in order to prevent the beam from twisting we have

$$Ve = \frac{Vb_1^2 h^2 t}{4I}.$$

But

$$I = 2b_1t\left(\frac{h}{2}\right)^2 + \frac{1}{12}wh^3 = \frac{1}{2}b_1th^2\left(1 + \frac{1}{6}\frac{wh}{b_1t}\right),$$

in which w is the thickness of the web. Hence,

$$e = \frac{\frac{1}{2}b_1}{1 + \frac{1}{6}\frac{wh}{b_1t}} = \frac{\frac{1}{2}b_1}{1 + \frac{1}{6}\frac{a_w}{a_f}}, \quad \ldots \quad (237)$$

in which a_w is the area of the web, and a_f is the area of one outstanding flange. This expression shows that e is large when the ratio of the area of the web to the area of the outstanding flange is relatively small, but

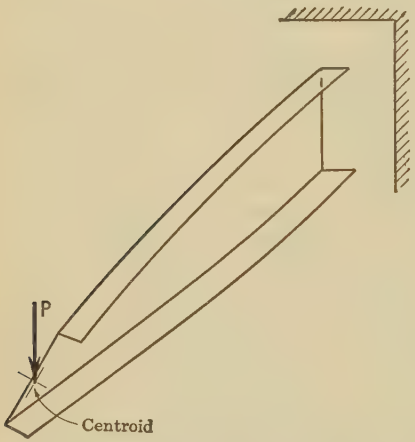

FIG. 294.—Twisting of channel due to transverse load through centroid.

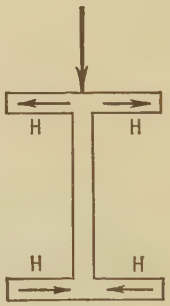

FIG. 295.—Lateral shear in flanges of I-section.

that the value of e is always less than one-half the width of the outstanding flange.

A channel beam, therefore, twists as it bends when the loads are applied through the centroid, as indicated in Fig. 294. Further, the maximum longitudinal stress developed may be largely in excess of that given by the flexure formula.

In the case of an I-section loaded through the centroid of the section, a lateral shear is developed in each outstanding flange just as in a channel section, but the shears in each flange are equal and opposite, as indicated in Fig. 295, and hence there is no resultant twisting moment to be balanced by the external forces.

The shear center for a T-section, or an angle section, or any section composed of two narrow rectangles, is the intersection of the center lines parallel to the long sides of the two narrow rectangles.

It will be observed that the twisting of the beam occurs only when the beam is so loaded that it is subjected to shear. For pure bending, therefore, the shear center has no significance.

135. Shear Center for Certain Other Sections Having One Axis of Symmetry.—Let it be required to find the position of the vertical plane of loading for the unequal-flanged H-section and the T-section shown in Fig. 296 so that the axis of symmetry will be the neutral axis, and the flexure formula will be applicable.

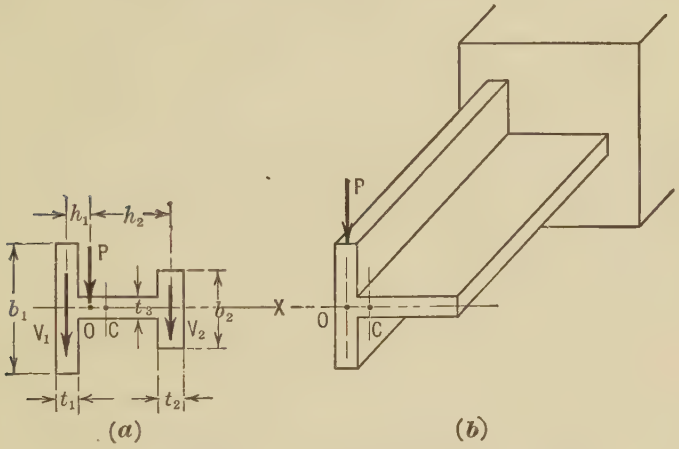

FIG. 296.—Shear center for unequal-flanged H-section.

A simple approximate solution applicable to relatively thin sections may be obtained as follows. In Fig. 296(*a*) the bending resistance offered by the *web* may be considered to be negligible, and hence the two flanges may be considered to be two separate beams. If each flange bends, without twisting, in accordance with the flexure formula with the axis of symmetry as the neutral axis, the radii of curvature, $R = \frac{EI_x}{M}$, of the two flanges at any section will be equal and hence

$$\frac{R}{E} = \frac{I'_x}{M_1} = \frac{I''_x}{M_2}.$$

But the bending moments M_1 and M_2 at any section along the beam are equal, respectively, to the product of the vertical shears V_1 and V_2 in the flanges and the distance to the section. Hence

$$\frac{I'_x}{V_1} = \frac{I''_x}{V_2}.$$

Further, since there is no twisting of the section, the load P is transferred to the flanges in producing V_1 and V_2 so that

$$V_1 h_1 = V_2 h_2.$$

Therefore,

$$\frac{h_1}{h_2} = \frac{I''_x}{I'_x}, \quad \ldots \ldots \ldots \quad (238)$$

in which h_1 and h_2 are the distances from the shear center of the section to the centroids of the flange cross-sections through which the shears V_1 and V_2 may be assumed to act provided the flanges are relatively thin.

Eq. (238) shows that the shear center, O, of the section lies between the centroid of the whole section and the centroid of the flange having the larger moment of inertia. In the limiting case in which there is only one flange, as in the T-section (Fig. 296*b*), the shear center is at the centroid of the flange if the bending resistance of the stem is assumed, as before, to be negligible.

PROBLEMS

432. A 6-in. 15.3-lb. ship channel is used as a horizontal simple beam. A uniformly distributed load is applied so that the beam bends with the axis of symmetry as the neutral axis. Determine the location of the plane of the load. See a steel maker's handbook for the dimensions of the channel. *Ans.* $e = 1.23$ in.

433. The section of Fig. 296(*a*) has the following dimensions: $t_1 = \frac{1}{2}$ in., $t_2 = 1$ in., $t_3 = \frac{1}{2}$ in., $b_1 = 6$ in., $b_2 = 4$ in., $h = 8$ in. Locate the shear center O through which P must act in order that the axis of symmetry shall be the neutral axis. Assume that $h = h_1 + h_2$. *Ans.* 1.35 in. from c.g.

434. An angle section with equal legs is used as a cantilever beam so that the axis of symmetry is the neutral axis. Draw a sketch of the beam showing where a vertical concentrated load at the free end should act.

435. An 8-in. by 8-in. by $\frac{1}{2}$-in. rolled-steel angle is used as a horizontal cantilever beam with a length of 6 ft. Find the safe vertical concentrated load applied at the free end (*a*) if the load lies in the axis of symmetry, (*b*) if the load acts so that the axis of symmetry is the neutral axis. Use a working unit-stress of 18,000 lb. per sq. in. See Appendix I for method of finding the moments of inertia.

Ans. (*a*) 1560 lb. (*b*) 3440 lb.

CHAPTER XV

UNSYMMETRICAL BENDING

136. Introduction.—One of the assumptions on which the flexure formula, $M = \frac{sI}{c}$, is based is that the neutral axis for any cross-section of the beam passes through the centroid of the section and is perpendicular to the plane of the loads. But, as shown in the next article, this condition is satisfied only when the plane of the loads contains one of the principal [1] axes of inertia passing through the shear center [2] of the section; such a plane is called a principal plane of bending. An axis of symmetry is always a principal axis, and bending caused by

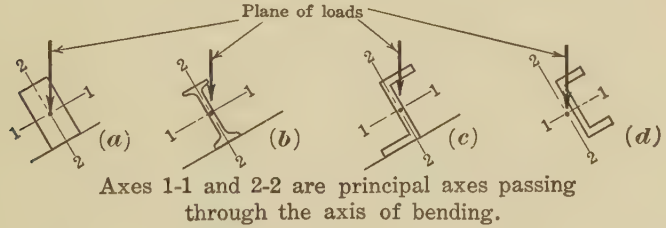

FIG. 297.—Unsymmetrical loading.

loads that do not lie in a principal plane of bending is usually called unsymmetrical bending.

Some beams, such as roof purlins, are subjected to loads whose planes make large angles with principal planes, as indicated in Fig. 297, whereas many beams in structures and machines are subjected to loads whose planes are inclined only slightly to principal planes.

Now the shape of most of the steel rolled-sections, such as I-sections, channel-sections, etc., that are used for beams is such as to give maximum resistance to bending *on the assumption* that the stresses developed

[1] For a discussion of principal axes see Appendix I.

[2] As discussed in the preceding chapter, when the plane of the loads contains an axis of symmetry the shear center and centroid both lie in the plane of the loads.

are in accordance with the flexure formula, $s = \frac{M}{c}$; in other words, the shape of the cross-section is made so that $\frac{I}{c}$ is as large as is practicable. It will be shown in this chapter, however, that a section which is well designed to resist bending due to symmetrical loading may be poorly designed to resist bending due to unsymmetrical loading, even though the plane of the loads deviates only slightly from a plane of symmetry.

137. Stress in Beam Subjected to Unsymmetrical Loading.—In the following discussion it is assumed that the plane of the loads contains

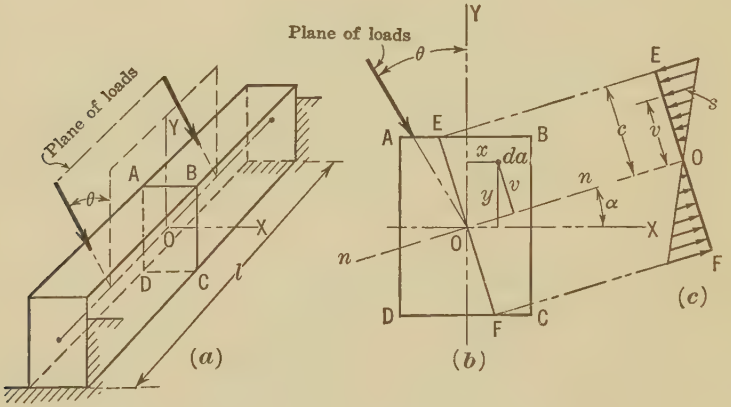

FIG. 298.—Stress in unsymmetrically loaded beam.

the bending axis of the beam, and hence the beam is not subjected to torsion; that the loads are perpendicular to the bending axis of the beam, and hence the bending is not accompanied by axial tension or compression; and that the shearing stresses in the beam are negligible

In Fig. 298(*a*) is represented a straight beam having a constant cross-section (assumed rectangular for convenience only) subjected to loads that lie in a plane making an angle θ with the Y-plane. Fig. 298(*b*) represents (enlarged) the cross-section $ABCD$ in the central portion of the beam. Let nn (Fig. 298*b*) represent the neutral axis and let α denote the angle between the neutral axis and the axis OX.

Let it be required to determine the direction of the neutral axis and the intensity of stress at any point in the cross-section; that is, let it be required to find the values of α and s.

It is assumed that, on any cross-section, the unit-stress varies directly as the distance from the neutral axis, provided that the proportional

limit of the material is not exceeded; the justification for this assumption is the same as in the case of symmetrical loading. Fig. 298(*c*), then, shows the way the unit-stress varies along any line *EF*, perpendicular to the neutral axis. Let a tensile stress be considered positive and a compressive stress negative; the stress, then, on the elementary area *da* (Fig. 298*b*) is negative. If *s* denotes the unit-stress at any point at the distance *v* from *nn*, then

$$s = -kv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (239)$$

where *k* is a constant. But since the position of the neutral axis is unknown, it will be convenient to express *v* in terms of rectangular coordinates with respect to arbitrarily selected axes. Thus, in Fig. 298(*b*) let *OX* and *OY* be such a pair of axes; from the geometry of the figure we have

$$v = (y - x \tan \alpha) \cos \alpha$$

$$= y \cos \alpha - x \sin \alpha$$

and hence eq. (239) becomes

$$s = -k(y \cos \alpha - x \sin \alpha). \quad . \quad . \quad . \quad . \quad (240)$$

There are three unknown quantities in this expression (*s*, *k*, and α), and hence two additional equations must be found before *s* and α can be determined; these are furnished by the conditions of equilibrium. Since the beam is in equilibrium, the sum of the moments of the external forces that lie on one side of the section *ABCD* about each of two rectangular axes in the section must be equal and opposite, respectively, to the sum of the moments of the stresses on the section about the same axes.

Let *M* denote the bending moment in the plane of the loads; then, the bending moment about the axis *OX* is $M_x = M \cos \theta$, and the bending moment about the axis *OY* is $M_y = M \sin \theta$. Now the moment of the stresses on the section *ABCD* about the axis *OX* is $\int sda\ y$. Hence taking moments about the *OX* axis we have, for equilibrium,

$$M \cos \theta = \int sda\, y$$

$$= \int -k(\cos \alpha\, y^2 da - \sin \alpha\, xyda)$$

$$= -k \cos \alpha \int y^2 da + k \sin \alpha \int xyda.$$

But $\int y^2 da$ is the moment of inertia of the cross-sectional area of the beam about the axis OX, and is denoted by I_x, and $\int xyda$ (denoted by P_{xy}), is the product of inertia of the same area with respect to the axes OX and OY (see Appendix I). However, the product of inertia is equal to zero with respect to principal axes. Therefore, *if the axes OX and OY are so selected that they are principal axes*[3] we have,

$$M \cos \theta = - k \cos \alpha I_x.$$

Similarly, if moments are taken about the axis OY, we have

$$M \sin \theta = \int sda\, x = - k \cos \alpha \int xyda + k \sin \alpha I_y;$$

and this for principal axes becomes

$$M \sin \theta = k \sin \alpha I_y.$$

Therefore, if θ, α, x and y refer to principal axes, we have

$$M \cos \theta = - k \cos \alpha I_x, \quad . \quad . \quad . \quad . \quad . \quad (241)$$

$$M \sin \theta = k \sin \alpha I_y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (242)$$

Direction of Neutral Axis.—By dividing eq. (242) by eq. (241) the equation found is

$$\text{Tan } \alpha = - \frac{I_x}{I_y} \tan \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (243)$$

from which the direction of the neutral axis may be found when the direction of the plane of the loads and the properties of the cross-section are known. The negative sign arises from the fact that θ is measured from one of the principal axis and α is measured (in the same direction) from the other principal axis, as shown in Fig. 298; as a rule, the sign may be neglected and the quadrant in which α lies may easily be found by inspection; but by giving the proper sign to tan θ the equation will give the proper quadrant for α.

Unit-stress at Any Point in Cross-section.—By solving eq. (241) for cos α and eq. (242) for sin α, and by substituting these values in eq. (240), the following expression for s is found:

$$s = \frac{M \cos \theta\, y}{I_x} + \frac{M \sin \theta\, x}{I_y} = \frac{M_x y}{I_x} + \frac{M_y x}{I_y}. \quad . \quad . \quad (244)$$

[3] Methods of selecting principal axes are discussed in Appendix I.

Eqs. (243) and (244) are valid only when α, θ, x, y, I_x and I_y refer to principal axes; these axes are frequently convenient axes to use, since axes of symmetry are principal axes and the properties of the commonly rolled sections with respect to axes of symmetry are given in handbooks.

Eq. (244) shows that the unit-stress at any point in the cross-section may be found as follows: Calculate the bending moment about each of the principal axes; then calculate the unit-stress due to each of these component moments separately, by use of the flexure formula, and add (algebraically) the unit-stresses thus found. The sign of each of the two parts in eq. (244) can usually be determined easily from inspection. It may be noted, however, that when x, y, and θ are used as in Fig. 298, s will be a tensile stress when positive (and a compressive stress when negative) if M_x and M_y are considered to be positive when they produce tensile stress (and negative when they produce compressive stress) in the portions of the section that lie on the positive sides of the axes.

Flexure Formula Requires that Plane of Loads Shall Contain Principal Axis; Neutral Axis is then Perpendicular to Plane of Loads.—If, in eq. (244), θ is made equal to zero (that is, if the plane of the loads contains a principal axis), $\cos\theta = 1$ and $\sin\theta = 0$, and hence eq. (244) becomes $s = \dfrac{My}{I_x}$, which is the ordinary flexure formula for symmetrical loading. And if, in eq. (243), θ is made equal to zero, α becomes equal to zero, and hence the neutral axis is coincident with one principal axis, and is, therefore, perpendicular to the plane of the loads which contains the other principal axis.

Combined Bending and Axial Loads.—If an axial load acts on the beam in addition to the unsymmetrical bending loads, the fiber unit-stress may be found by adding (algebraically) the unit-stress caused by the axial load as if acting alone and the unit-stress due to the bending loads as if acting alone.[4]

ILLUSTRATIVE PROBLEMS

Problem 436. The rectangular beam shown in Fig. 298(*a*) and 299 is 6 in. wide and 8 in. deep; it is used as a simple beam on a span of 16 ft. Two loads of 800 lb. each are applied to the beam, each load being 4 ft. from an end of the span; the plane of the loads makes an angle of 30° with the vertical plane of symmetry. Find the direction of the neutral axis and the fiber unit-stresses at A.

[4] This method is frequently called the principle of superposition, see last paragraph in Art. 43.

Solution.—

$$I_x = \tfrac{1}{12}bh^3 = \tfrac{1}{12}6(8)^3 = 256 \text{ in.}^4;\quad I_y = \tfrac{1}{12}hb^3 = 144 \text{ in.}^4$$

$$\text{Tan } \alpha = \frac{I_x}{I_y}\tan\theta = \tfrac{256}{144} \times 0.577 = 1.028.$$

Hence

$$\alpha = 45^\circ\,51',\text{ as shown in Fig. 299.}$$

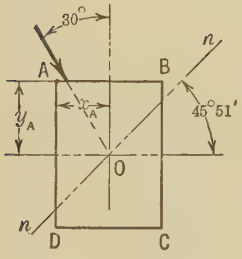

FIG. 299.

All points in the area above the neutral axis nn (Fig. 299) are subjected to compressive stresses, and all points below to tensile stresses. The unit-stress at the corner A, which is the maximum unit-stress in the beam, is

$$s_A = \frac{M_x y_A}{I_x} + \frac{M_y x_A}{I_y}.$$

It is evident from inspection that each component moment causes compressive stress at A, and hence s_A is found by adding the two parts of the right-hand member of the equation; however, the signs of x, y, M_x and M_y will cause the equation to yield the correct results. Thus, $x_A = -3$ in., $y_A = +4$ in., $M_x = -(800 \times 4 \times 12 \times 0.866) = -33{,}300$ lb.-in., and $M_y = +(38{,}400 \times 0.5) = +19{,}200$ lb.-in. Therefore,

$$s_A = \frac{-33{,}300 \times 4}{256} + \frac{+19{,}200 \times (-3)}{144}$$

$$= -518 - 400 = -918 \text{ lb. per sq. in. (compression).}$$

Problem 437. An 8-in. by 8-in. by $\frac{3}{4}$-in. angle is loaded as shown in Fig. 300(a), the total load P being 10,000 lb. Find the direction of the neutral axis. Also cal-

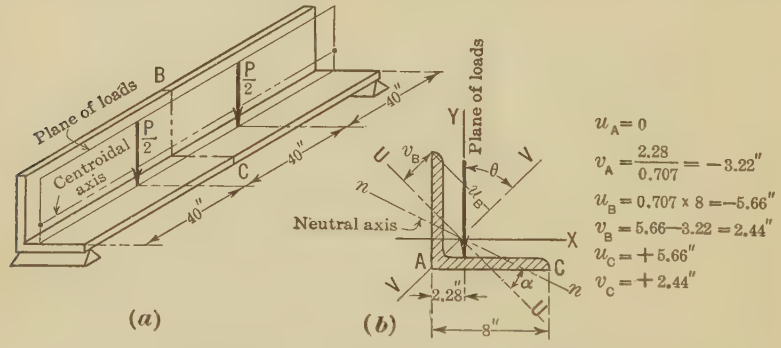

FIG. 300.—Unsymmetrically loaded angle-beam.

culate the values of the fiber unit-stresses at A, B, and C; compare these values with those found by assuming that the neutral axis is perpendicular to the plane of the loads and using the ordinary flexure formula. Since the central third of the beam is subjected to pure bending, the shear center and bending axis have no significance for this portion of the beam.

Solution.—The u- and v-axes (Fig. 300(b)) are the principal axes since the v-axis is an axis of symmetry. The following values are obtained from a steel-maker's handbook.

$$a = 11.44 \text{ sq. in.}, \quad x_A = y_A = 2.28 \text{ in.}, \quad I_x = I_y = 69.7 \text{ in.}^4, \quad r_u = 1.57 \text{ in.}$$

Hence,

$$I_u = ar_u^2 = 11.44 \times (1.57)^2 = 28.2 \text{ in.}^4$$

Also,

$$I_u + I_v = I_x + I_y \text{ (see App. I).}$$

Hence,

$$I_v = 2(69.7) - 28.2 = 111.2 \text{ in.}^4$$

Therefore,

$$\tan \alpha = \frac{I_u}{I_v} \tan \theta = \frac{28.2}{111.2} \tan 45° = \frac{28.2}{111.2} \times 1 = 0.254,$$

$$\alpha = 14° 20', \text{ as shown in Fig. 300}(b).$$

The central portion of the beam is subjected to a bending couple having a moment of $M = 5000 \times 40 = 200{,}000$ lb.-in. Further, the magnitude of M_u and M_v is

$$M_u = M_v = 200{,}000 \times 0.707 = 141{,}400 \text{ lb.-in.}$$

Unit-stress at A, B, and C

$$s_A = \frac{M_u v_A}{I_u} + \frac{M_v u_A}{I_v}$$

$$= \frac{-(141{,}400) \times (-3.22)}{28.2} = 16{,}200 \text{ lb. per sq. in., tension.}$$

$$s_B = \frac{-(141{,}000) \times 2.44}{28.2} + \frac{(141{,}000) \times (-5.66)}{111.2}$$

$$= -19{,}300 \text{ lb. per sq. in., compression.}$$

$$s_C = \frac{-(141{,}000) \times 2.44}{28.2} + \frac{(141{,}000) \times (5.66)}{111.2}$$

$$= -5100 \text{ lb. per sq. in., compression.}$$

The values for s at A, B and C as found from the flexure formula, assuming (erroneously) that the x-axis is the neutral axis, are:

$$s_A = s_C = \frac{My_A}{I_x} = \frac{200{,}000 \times 2.28}{69.7} = 6550 \text{ lb. per sq. in., tension.}$$

$$s_B = \frac{200{,}000 \times 5.72}{69.7} = 16{,}400 \text{ lb. per sq. in., compression.}$$

Since the unit-stress at B would govern the allowable moment or load, the value of the allowable load for this angle beam as found from the flexure formula assuming the x-axis to be the neutral axis would be about 15 per cent too large.

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438. A timber beam 10 in. wide by 12 in. deep by 14 ft. long is used as a simple beam on a span of 12 ft. It is subjected to a concentrated load of 6000 lb. at the mid-section of the span. If the plane of the load makes an angle of 45° with the vertical axis of symmetry (similar to θ in Fig. 298(a)), find the direction of the neutral axis and the maximum unit-stress in the beam.

Ans. $\alpha = 55° 15'$, $s = 1400$ lb. per sq. in.

FIG. 301.

439. The angle section 8-in. × 8-in. × $\frac{3}{4}$-in. shown in Fig. 300 is used as a horizontal cantilever beam 50 in. long and is subjected to a vertical concentrated load of 4000 lb. acting through the shear center of the end section as shown in Fig. 301 (the beam therefore bends without twisting, see Art. 135). Calculate the stress at A, B and C in the section at the fixed end of the beam.

Ans. $s_B = +19{,}400$, $s_A = -16{,}200$, $s_C = +5130$ lb. per sq. in.

440. A 6-in. 8.2-lb. channel is used as a purlin for a roof (Fig. 297(d)). If the pitch of the roof is $\frac{1}{4}$ (that is, the roof rises $\frac{1}{2}$ ft. per horizontal foot, thus the total rise is $\frac{1}{4}$ of the span) and the span is 10 ft., find the maximum unit-stress caused by a uniformly distributed vertical load of 100 lb. per ft. of span; the plane of the load contains the bending axis of the beam.

Ans. $s_A = -16{,}500$ lb. per sq. in.

138. Change in Slope of Neutral Axis and Increase in Stress in Rolled Sections Due to a Very Small Inclination of Plane of Loads to a Principal Plane.—Eq. (243) shows that α is large even though θ is relatively small, provided that the cross-section of the beam is of such a shape that I_x is much larger than I_y. Thus the neutral axis of I-beams, channels, etc., is steeply inclined to the horizontal axis of symmetry when the plane of the loads deviates but little from the vertical plane of symmetry, and likewise the maximum fiber stress is increased considerably, as is indicated in Table 8 in which the plane of the loads is assumed to make an angle of one degree with a principal plane.

Table 8 indicates clearly that a section that is well proportioned to resist bending due to loads in a plane of symmetry, such as I-sections, may be poorly designed to resist bending due to loads in a plane that makes only a very small angle with the plane of symmetry.

139. Eccentric Load Not in Plane of Symmetry.—In Art. 64 the action line of the resultant longitudinal load passed through a point on an axis of symmetry of the area on which the stress was desired.

If the load P (Fig. 303a) does not lie in a plane containing an axis of symmetry it may be resolved into an axial load P_1 (equal to P) and a couple, P and P_2, having a moment Pe, and this couple may be further resolved into two component couples in planes containing the axes of symmetry or principal axes, OX and OY; the stress due to each component couple may then be found from the ordinary flexure

TABLE 8

PERCENTAGE INCREASE IN FIBER UNIT-STRESS AND CHANGE IN SLOPE OF THE NEUTRAL AXIS, CAUSED BY A ONE-DEGREE INCLINATION OF THE PLANE OF THE LOADS TO A PRINCIPAL PLANE

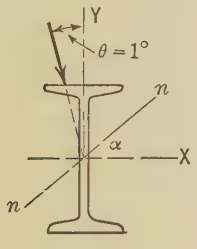
Legend	I-beam	$\frac{I_x}{I_y}$	Increase in Fiber Stress (per cent)	Change in Slope of Neutral Axis
	27-in. 90-lb.	39.3	23.0	34° 20′
	20-in. 65-lb.	41.8	22.8	36° 10′
	18-in. 55-lb.	37.5	21.8	33° 15′
	15-in. 42-lb.	30.2	19.3	27° 15′
	12-in. 31½-lb.	22.7	16.5	21° 35′
	10-in. 25-lb.	17.7	14.4	17° 14′
	8-in. 18-lb.	15.0	13.1	14° 40′
	6-in. 12¼-lb.	11.8	11.5	11° 40′

FIG. 302.

formula as in the preceding article. Thus, the moment of the component couple in the plane containing the axis OX is $Pe \cos \theta$ or Pe_x (Fig. 303a), the forces of this couple being represented by P_2 and P_3 in Fig. 303(b), and the moment of the component couple in the plane containing OY is Pe_y and the forces of this couple are P_4 and P_5.

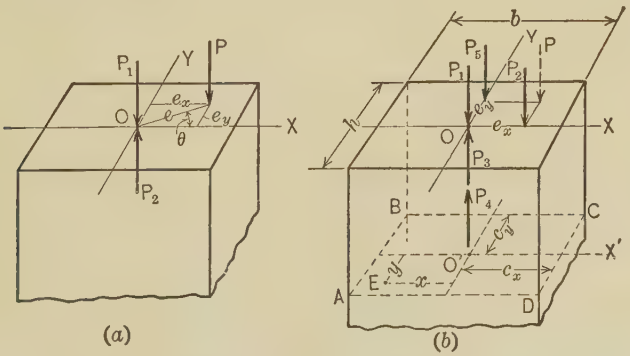


FIG. 303.—Eccentric load not in plane of symmetry.

These five forces, each equal to P, will produce the same unit-stress at any point in a section (such as the section $ABCD$) as the original force P, and the normal unit-stress at any point in the section is the algebraic sum of the unit-stresses produced by the axial load P_1 and the two bending couples P_2, P_3, and P_4, P_5.

The maximum unit-stress occurs at C and is

$$s = s_1 + s_2 + s_3 = \frac{P}{a} + \frac{Pe_x c_x}{I_y} + \frac{Pe_y c_y}{I_x}.$$

For other points in the area a either the second or the third term or both may be negative. Thus the unit-stress at a point E in the quadrant $O'A$ (Fig. 303*b*) is

$$s = \frac{P}{a} - \frac{Pe_x x}{I_y} - \frac{Pe_y y}{I_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (245)$$

Kern of a Section.—It is evident that the least value of s will occur at A and its value will be given by equation (245) when $x = \frac{b}{2}$ and $y = \frac{h}{2}$, and it will be a tensile stress if the sum of the last two terms is greater than $\frac{P}{a}$. If a tensile stress is to be avoided therefore, the values of e_x and e_y cannot be greater than those found from the equation

$$\frac{Pe_x\frac{b}{2}}{I_y} + \frac{Pe_y\frac{h}{2}}{I_x} = \frac{P}{a}, \quad \text{or} \quad \frac{e_x\frac{b}{2}}{\frac{1}{12}hb^3} + \frac{e_y\frac{h}{2}}{\frac{1}{12}bh^3} = \frac{1}{bh}$$

whence,

$$\frac{e_x}{b} + \frac{e_y}{h} = \frac{1}{6} \quad \text{or} \quad \frac{e_x}{\frac{1}{6}b} + \frac{e_y}{\frac{1}{6}h} = 1,$$

which is the equation of a straight line that intersects the axis OX at a distance $\frac{b}{6}$ from O, and the axis OY at a distance $\frac{h}{6}$ from O (Fig. 304).

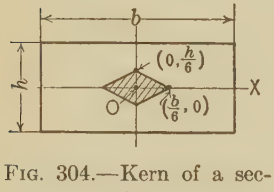


FIG. 304.—Kern of a section.

Similar limits occur in the other quadrants and hence the resultant load on the member must pass within the shaded area shown in Fig. 304 if tensile stress in the member is to be avoided.

The area within which the resultant load must pass to avoid tensile stress on a section is frequently called the *core*, *kernel*, or *kern* of the section. Thus for a *rectangular section* the kern is a rhombus the diagonals of which are the middle-thirds of the principal axes of the section.

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441. A short rectangular timber having a section 6 in. by 8 in. is subjected to a longitudinal eccentric compressive load of 10,000 lb. The action line of the load passes through a point in the section 3 in. from the 6-in. side and $2\frac{1}{2}$ in. from the 8-in. side. Find the unit-stress at each corner.

Ans. s_A = 52 lb. per sq. in. T., s_B = 156 lb. per sq. in. C.
s_C = 468 lb. per sq. in. C., s_D = 260 lb. per sq. in. C.

442. The resultant normal pressure, P, on top of a concrete base having a square cross-section, a, acts at the center of one quadrant of the square. Will tensile stresses occur in the base? If so, find the maximum tensile and compressive unit-stresses that will be developed, assuming the values of P and a to be 80,000 lb. and 4 sq. ft., respectively.

CHAPTER XVI

CURVED FLEXURAL MEMBERS

140. Introduction.—One of the assumptions on which the flexure formula, $M = \frac{sI}{c}$, is based is that the member (beam) to which the bending moment M is applied is straight before the beam is bent; the above formula, therefore, will be referred to as the *straight*-beam formula.[1] Many members that are subjected to bending, however, are curved before a bending moment is applied to them, as, for example, crane hooks, chain links, frames of punching machines, etc. Such members will be called curved beams. Further, some members have curved portions that are subjected to bending even though the members as a whole are not flexural members, for example, the portion in the eye of an eye bar in a bridge, etc. It is important therefore to determine the effect of initial curvature of a beam on the stress in the beam.

141. Essential Difference between Straight Beams and Curved Beams.—In the following discussion it will be assumed that all the conditions required to make the straight-beam formula applicable are satisfied except that the beam is curved. Let the curved beam DOE (Fig. 305) be subjected to the loads (and reactions) Q. There is one surface in the beam in which the fibers do not change in length (this surface is called the neutral surface), and the assumption is made that the total deformations (total strains) of the fibers vary directly as the distances of the fibers from the neutral surface, shortening above the neutral surface and stretching below the neutral surface, or, as frequently stated, a plane section before bending is assumed to be a plane section after bending. This assumption of conservation of a plane section is verified approximately by direct measurements of the strains by means of a strain gage. Thus in Fig. 305 consider the fibers between the two lines AB and A_1B_1; these lines are considered to be the traces of two normal

[1] It will be found, however, that this so-called straight-beam formula applies with very little error, to beams that would normally be thought of as curved; the term "curved beam," then, usually means a flexural member with a *relatively sharp* curvature.

sections of the beam before the loads are applied. After the beam is loaded, the change in the length of any fiber is represented by the distance along the fiber between the lines A_1B_1 and $A'_1B'_1$; the neutral surface is represented by NN_1, and the stretch of fiber PP_1 is $P_1P'_1$, etc.

Although the total strains of the fibers are proportional to the distances of the fibers from the neutral surface, the unit-strains of the fibers are not proportional to the distances because the fibers are not of equal length, whereas in a straight beam the fibers *are* of equal length, and hence the unit-strains in a straight beam, as well as the total strains, are proportional to the distances of the fibers from the neutral axis. Now, if the stress in a beam does not exceed the propor-

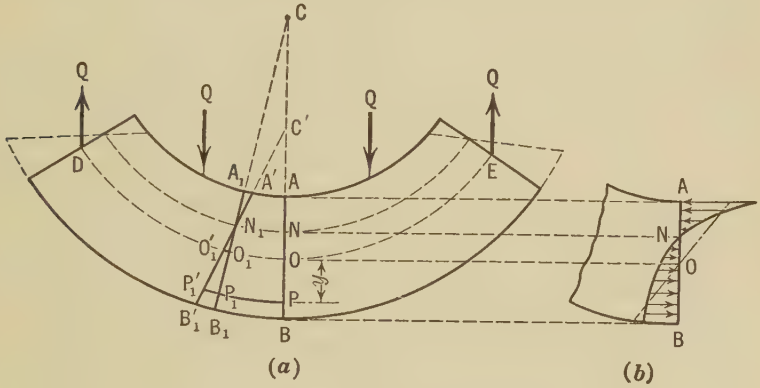


FIG. 305.—Curved beam.

tional limit of the material, the unit-stress on any fiber is proportional to the unit-strain of the fiber; therefore, the unit-stresses on the fiber of a curved beam are *not* proportional to the distances of the fibers from the neutral surface. It follows, then, that the resisting moment in a curved beam is not given by the expression $\frac{sI}{c}$, since this is obtained on the assumption that the unit-stress varies directly as the distance from the neutral axis. For the same reason, the neutral axis in a curved beam does not pass through the centroid of the section; the position of the neutral axis must be such that the sum of the compressive forces on the fibers at any section is equal to the sum of the tensile forces, and hence the neutral axis must shift from the centroidal axis towards the side having the higher stresses.

The way the intensity of stress varies over the section and the relative position of the neutral axis is shown in Fig. 305(*b*); if the beam were

straight, the stress would be zero at the centroidal axis and would vary directly with the distance from the centroidal axis as indicated by the dotted line in Fig. 305(*b*). Quantitative results are found in the next article.

142. Unit-stress at Any Point in a Curved Beam. The Winkler-Bach Formula.—Let it be required to find the unit-stress at any point in a cross-section of a curved beam in terms of the external forces (bending moment) and the dimensions of the sections. The unit-stress s on a fiber PP_1 (Fig. 306) at any distance y from the centroidal axis is

$$s = E\varepsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (246)$$

in which E is the modulus of elasticity of the material and ε is the unit-strain of the fiber. Now if ε can be expressed in terms of the bending moment M and the dimensions of the cross-section, then this value of ε may be substituted in (246) and the desired expression for s found. It will be found convenient to express ε of any fiber in terms of the unit-strain (ε_0) of the centroidal fiber, and the angle of rotation of the section.

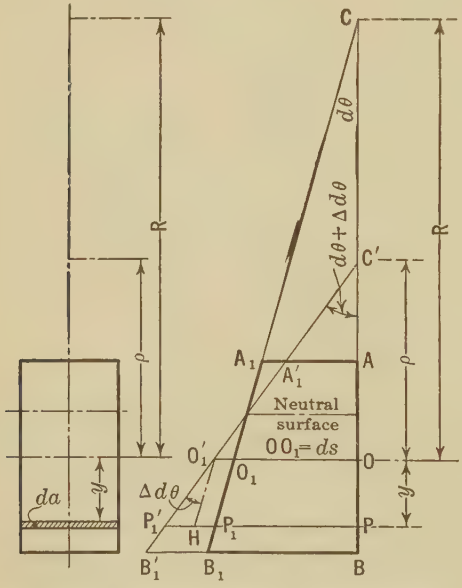


FIG. 306.—Enlarged section of curved beam.

Let Fig. 306 represent the part ABB_1A_1 of Fig. 305(*a*), enlarged; the angle between the two sections AB and A_1B_1 is $d\theta$. The bending moment causes the plane A_1B_1 to rotate through an angle $\Delta d\theta$, thereby changing the angle this plane makes with the plane BAC from $d\theta$ to $(d\theta + \Delta d\theta)$; the center of curvature is changed from C to C', and the distance of the centroidal axis from the center of curvature is changed from R to ρ. It should be noted that y, R and ρ at any section are measured from the centroidal axis and not from the neutral axis.

Let ds denote the length of the centroidal fiber OO_1, and ε_0 the unit-strain of this fiber; then since the two planes AB and A_1B_1 are only a differential distance apart

$$\varepsilon_0 = \frac{O_1O'_1}{OO_1} \quad \text{or} \quad O_1O'_1 = \varepsilon_0 ds = \varepsilon_0 R d\vartheta$$

and

$$\varepsilon = \frac{P_1P'_1}{PP_1} = \frac{P_1H + HP'_1}{PP_1} = \frac{O_1O'_1 + HP'_1}{PP_1}.$$

But, $O_1O'_1 = \varepsilon_0 R d\theta$; $HP'_1 = O'_1H \cdot \Delta d\theta = y\Delta d\theta$; $PP_1 = (R + y)d\theta$.

Therefore

$$\varepsilon = \frac{\varepsilon_0 R d\theta + y\Delta d\theta}{(R + y)d\theta} = \frac{\varepsilon_0 R + y\dfrac{\Delta d\theta}{d\theta}}{R + y}.$$

For convenience, let the angular unit-strain, $\dfrac{\Delta d\theta}{d\theta}$, be denoted by ω. By adding and subtracting $\varepsilon_0 y$ to the numerator, the above expression may be transformed to the following equation:

$$\varepsilon = \varepsilon_0 + (\omega - \varepsilon_0)\frac{y}{R + y}.$$

From eq. (246), then,

$$s = E\varepsilon = E\left[\varepsilon_0 + (\omega - \varepsilon_0)\frac{y}{R + y}\right]. \quad . \quad . \quad . \quad . \quad (247)$$

Since there are three unknown quantities (s, ω, and ε_0) in this equation, two additional equations must be found. These are obtained from the conditions of equilibrium; namely (1) the algebraic sum of the moments of the external forces (bending moment) about any axis in the cross-section is equal to the moment of the stresses on the section about the same axis, that is, the bending moment M is equal to the resisting moment $\int y \cdot sda$, and (2) the algebraic sum of all the normal (longitudinal) stresses on the cross-section is equal to zero, that is, $\int sda = 0$. Thus we have,

$$M = \int y \cdot sda = \int Ey\left[\varepsilon_0 + (\omega - \varepsilon_0)\frac{y}{R + y}\right]da \quad . \quad (248)$$

and

$$\int sda = \int E\left[\varepsilon_0 + (\omega - \varepsilon_0)\frac{y}{R + y}\right]da = 0. \quad . \quad . \quad . \quad . \quad (249)$$

And if the modulus of elasticity, E, has the same value for all the

fibers, it is a constant in the equation, and hence eqs. (248) and (249) may be written

$$M = E\left[\varepsilon_0 \int y da + (\omega - \varepsilon_0) \int \frac{y^2}{R+y} da\right] \quad . \quad . \quad . \quad (250)$$

and

$$\varepsilon_0 \int da = -(\omega - \varepsilon_0) \int \frac{y}{R+y} da. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (251)$$

But $\int da = a$, and since y is measured from the centroidal axis, $\int y da = 0$. Further for convenience, let

$$\int \frac{y}{R+y} da = -Za, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (252)$$

in which Z is a property of the area somewhat similar to the moment of inertia in the straight-beam formula; values of Z for circular and rectangular areas are found in Appendix I. The integral in eq. (250) may also be expressed in terms of Z as follows:

$$\int \frac{y^2}{R+y} da = \int \left(y - R\frac{y}{R+y}\right) da = -R \int \frac{y}{R+y} da = ZaR.$$

Eqs. (250) and (251) now become

$$M = E(\omega - \varepsilon_0) ZaR$$

$$\varepsilon_0 = (\omega - \varepsilon_0) Z.$$

Hence

$$(\omega - \varepsilon_0) = \frac{M}{EZaR}, \; \varepsilon_0 = \frac{M}{EaR} \text{ and } \omega = \frac{1}{Ea}\left(\frac{M}{R} + \frac{M}{RZ}\right)$$

Substituting these values in Eq. (247) we have

$$s = \frac{M}{aR}\left(1 + \frac{1}{Z}\frac{y}{R+y}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (253)$$

in which s is the unit-stress at a point at the distance y from the centroidal axis of the section; M is the bending moment at the given section, and R is the distance from the centroidal axis of the section to the center of curvature of the central axis of the unstressed beam; a is the area of the cross-section and Z is a property of the cross-section defined above, the values of which for circular and rectangular sections are given in Appendix I. Eq. (253) is frequently referred to as the Winkler-Bach formula.

Signs.—The bending moment M is positive when it increases curvature and negative when it decreases curvature; y is positive when measured toward the convex side of the beam and negative when measured toward the concave side, that is, toward the center of curvature.

Location of Neutral Axis.—According to eq. (253) the unit-stress on any fiber at the distance y from the centroidal axis, due to the bending moment M, is

$$s = \frac{M}{aR}\left(1 + \frac{1}{Z}\frac{y}{R+y}\right).$$

Now the stress at the neutral axis is zero; hence if y_0 denotes the distance of the neutral axis from the centroidal axis, the value of y_0 may be found by equating s, in the above equation, to zero. Thus,

$$1 + \frac{1}{Z}\frac{y_0}{R+y_0} = 0$$

$$y_0 = -\frac{ZR}{Z+1}. \quad . \quad . \quad . \quad . \quad . \quad (254)$$

As noted above, the negative sign means that y_0 is measured from the centroidal axis towards the center of curvature. It is important to note that eq. (254) applies to a beam that is subjected to bending only.

Axial and Bending Loads Combined.—If a curved beam is subjected to a normal load P acting through the centroid of the area of the section (axial load) in addition to a bending moment, the resulting unit-stress is the algebraic sum of the unit-stresses caused by the axial load and bending loads. Thus the maximum unit-stress is

$$s = \frac{P}{a} + \frac{M}{aR}\left(1 + \frac{1}{Z}\frac{y}{R+y}\right). \quad . \quad . \quad . \quad . \quad (255)$$

The signs of M and y are determined as stated above, and the sign of P is plus when it produces tensile stress and negative when it produces compressive stress.

143. Correction Factors for Use in the Straight-Beam Formula.—The Winkler-Bach formula derived in the preceding article, although satisfactory theoretically, is limited in its usefulness because of the difficulty in evaluating the property, Z, of the area. The expression for this property of the cross-section is too complicated for practical use for all sections except perhaps the circle and the rectangle. Further, graphical methods give results that are likely to be considerably in error, particularly since the value of Z is always small, rarely exceeding 0.8 for a relatively sharp curvature and being less than 0.005 for a relatively large curvature.

It is desirable, therefore, to have available correction factors for use in the straight-beam formula for a variety of sections likely to be used in curved flexural members. If K denotes such a correction factor, the stress at the extreme fiber of a curved beam would be given by

$$s = K\frac{Mc}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (256)$$

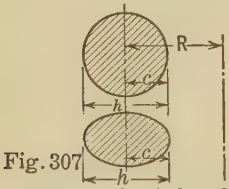
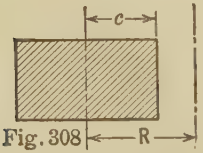
for pure bending, and

$$s = s_1 + K\frac{Mc}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

TABLE 9

CORRECTION FACTORS FOR USE IN FLEXURE FORMULA WHEN APPLIED TO CURVED BEAMS

Values of K in the Equation $s = K\frac{Mc}{I}$

Shape of Section	$\frac{R}{c}$	Factor by which s in flexure formula $\left(s = \frac{Mc}{I}\right)$ is multiplied to obtain unit-stress in curved beam: Values of K	
		At Inner Fiber	At Outer Fiber
 Fig. 307 K the same for circle and ellipse and independent of dimensions.	1.2	3.41	0.54
	1.4	2.40	0.60
	1.6	1.96	0.65
	1.8	1.75	0.68
	2.0	1.62	0.71
	3.0	1.33	0.79
	4.0	1.23	0.84
	6.0	1.14	0.89
	8.0	1.10	0.91
	10.0	1.08	0.93
 Fig. 308 K independent of section dimensions	1.2	2.89	0.57
	1.4	2.13	0.63
	1.6	1.79	0.67
	1.8	1.63	0.70
	2.0	1.52	0.73
	3.0	1.30	0.81
	4.0	1.20	0.85
	6.0	1.12	0.90
	8.0	1.09	0.92
	10.0	1.07	0.94

for combined direct stress and bending where

$$K = \frac{\dfrac{M}{aR}\left(1 + \dfrac{1}{Z}\dfrac{c}{R+c}\right)}{\dfrac{Mc}{I}}. \quad . \quad . \quad . \quad . \quad . \quad (258)$$

Values of correction factors for the inside fiber A and for the outside fiber B for circular and rectangular sections are given in Table 9. These values may be used for determining approximate but reliable values of stress in a curved beam of nearly any shape of section and any degree of curvature; the values of K do not vary greatly with different shapes of section. It is important to note that the values of K in Table 9 should be used in calculating the stress only in the extreme fibers of the curved beam. But, in beams free from abrupt changes in section, the maximum stress occurs at an extreme fiber and is usually the stress desired.

ILLUSTRATIVE PROBLEM

Problem 443. In Fig. 309 is shown a crane hook having a capacity of 10 tons. Fig. 309(*b*) gives the approximate dimensions of the cross-section at AB of Fig. 309(*a*). Calculate the unit-stress on the inner fiber and on the outer fiber (at A and B) when the hook is resisting its maximum load P of 10 tons.

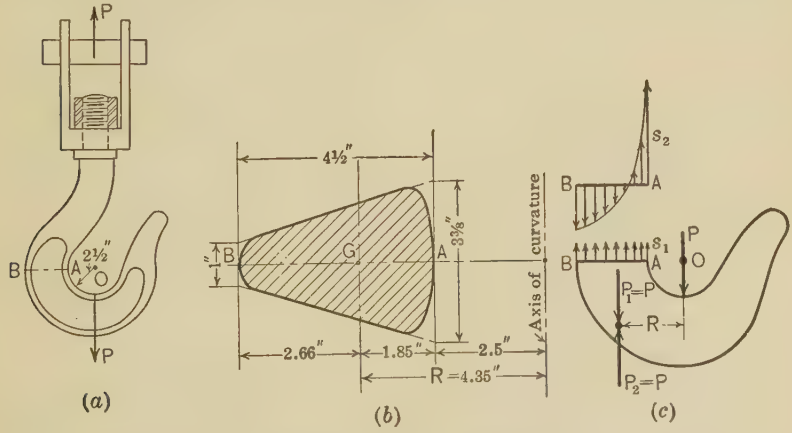


FIG. 309.—Stress in crane hook.

Solution.—The area of the section on which the maximum stresses occur (assuming the area to be a trapezoid) is, $a = 9.85$ sq. in. The centroid, G, is found (see Art. 166) to be $c_1 = 1.85$ in. from the inner fiber, Fig. 309(*b*). The radius of curvature then is, $R = 2.5 + 1.85 = 4.35$ in. The moment of inertia of the area about the centroidal axis is found (see Art. 172) to be $\bar{I} = 15.0$ in.^{4} The ratio of $\dfrac{R}{c} = \dfrac{4.35}{2.25} = 1.93$, and

hence the correction factors (using the values for a rectangular section) are: 1.55 for the inner fiber, and 0.72 for the outer fiber.

As indicated in Fig. 309(*c*), the load P may be resolved into an axial load $P_1 = P$ and a bending couple $M = PR$. The unit-stress, s_A, on the inner fiber is

$$s_A = s_1 + s_2 = \frac{P}{a} + 1.55\frac{Mc}{I} = \frac{10}{9.85} + \frac{1.55 \times 10 \times 4.35 \times 1.85}{15.0}$$

$$= 1.02 + 8.32 = 9.34 \text{ tons per sq. in.} = 18{,}700 \text{ lb. per sq. in., tensile stress.}$$

$$s_B = 1.02 - \frac{0.72 \times 43.5 \times 2.66}{15.0} = 1.02 - 5.55 = -4.53 \text{ tons per sq. in.}$$

$$= -9060 \text{ lb. per sq. in., compressive stress.}$$

If the correction factors were not used, that is, if the stress due to the bending moment were found by the straight-beam (flexure) formula the values would be

$$s_A = 12{,}800 \text{ lb. per sq. in., tension;}$$

$$s_B = 13{,}400 \text{ lb. per sq. in., compression.}$$

The maximum stress in the hook (at A) is, therefore, about 30 per cent greater than the value for s obtained by using the straight-beam formula.

PROBLEMS

444. The load P applied to the frame shown in Fig. 310 is 400 lb. Calculate the unit-stress at A and at B, and compare the values with those found by use of the straight-beam formula. The area at section AB is shown as the cross-hatched area.

Ans. $s_A = 15{,}100$ lb. per sq. in.

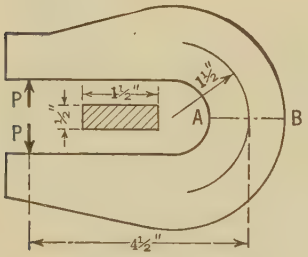


FIG. 310.

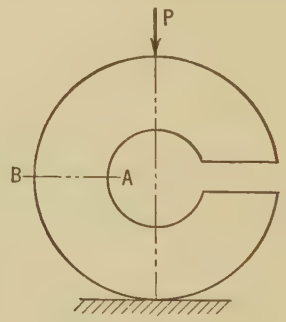


FIG. 311.

445. A ring (Fig. 311) is made of stock with a circular cross-section 3 in. in diameter. The inside diameter of the ring is 4 in. The load P is 4000 lb. Calculate the unit-stress at A and at B, and compare the values with those found by use of the straight-beam formula.

Ans. $s_A = 8240$ lb. per sq. in.

CHAPTER XVII

FLAT PLATES

144. Introduction.—A flat plate such as a cylinder head, bottom of a tank, floor panel, bulkhead, street man-hole cover, etc., when supported around the edge and subjected to loads normal to the surface, bends, but the bending differs from that of a beam in that the plate bends in all vertical planes whereas the beam may be assumed to bend in one plane only. Further, the bending of the plate in one vertical plane influences markedly the bending in all other vertical planes and hence the analysis of the stresses in a flat plate is much more involved than is that in a simple beam.

145. Stress in Circular Plate.—*Supported at Edge; Load Uniformly Distributed.*—Let a circular flat plate of radius r, and of constant thickness, t, be subjected to a uniformly distributed load of w per unit-area, and let the plate be supported around its edge by resting on a circular rim.

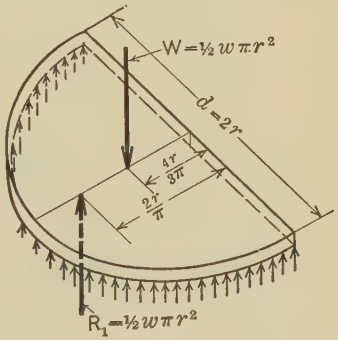


Fig. 312.—Forces acting on circular flat plate.

The bending moment, about any diameter, of the forces that lie to one side of the vertical diametral plane (Fig. 312) is the moment of the reaction of the supporting rim minus the moment of the downward load. The magnitude of the load on the half-plate is

$$W = w \cdot \tfrac{1}{2}\pi r^2$$

and its action line passes through the centroid of the semicircular area; this centroid is at the distance $\frac{4r}{3\pi}$ from the diametral plane (see App. I). The resultant R_1 of the reaction of the supporting rim must be equal in magnitude to the load, but its action line passes through the centroid of the semi-circumference, and hence acts at the distance $\frac{2r}{\pi}$ (see App. I) from the diametral plane as shown in Fig. 312.

Therefore, the bending moment, M, about the diameter is

$$M = R_1 \frac{2r}{\pi} - W \frac{4r}{3\pi} = \frac{w\pi r^2}{2}\left(\frac{2r}{\pi} - \frac{4r}{3\pi}\right) = \frac{wr^3}{3}.$$

Now the bending moment may be equated to the resisting moment at the diametral section since the resisting moment holds the bending moment in equilibrium. But the expression for the resisting moment is unknown. If the expression for the resisting moment is assumed to be the same $\left(\frac{sI}{c}\right)$ as that in a beam which bends in one plane only, that is, if the assumption is made that the stress at any point along a diameter is independent of the distance of the point from the center of the plate, we may write

$$\frac{wr^3}{3} = \frac{sI}{c}.$$

But $I = \frac{1}{12}(2r)t^3$, and $c = \frac{t}{2}$. Therefore, this equation becomes

$$s = w\left(\frac{r}{t}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (259)$$

The value of s in this equation is the average bending unit-stress at the surface of the plate at the diametral section. The maximum unit-stress in the plate is somewhat greater than the value of s in the above equation, but experiments by Bach and others indicate that eq. (259) gives a reliable value of the significant stress, particularly if the plate is made of ductile material and is subjected to a static load.

Circular Plates with Fixed Edges.—If a circular plate is rigidly held from bending (fixed) at the edge, the average bending moment (and average stress) at any diametral section is less than that given by the equations in the above article, because the negative bending moment at the edge decreases the positive moment within the central portion of the plate in much the same way as in a beam that is fixed at its ends; and also as in the case of a beam with fixed ends, the negative moment at the edge is usually greater than the moment at the center. The bending unit-stress at the fixed edge is $s = \frac{3}{4} w\left(\frac{r}{t}\right)^2$.

But, as plates are generally used, the edges are seldom "fixed" although there usually is some restraint; further, a slight amount of local yielding at the fixed edge may destroy much of the restraint and thereby transfer the moment to the central part of the plate.

For these reasons, the restraint at the edges of a plate is considered of less importance, particularly if the plate is made of relatively ductile material, than would be indicated by the results of mathematical analysis based on the assumption of fixed edges.

In general, a plate with a restrained (so-called fixed) edge will be intermediate in strength between the plate with a simply supported edge and the plate with an ideally fixed edge.

146. Stress in Square Plate. *Supported along Four Edges; Load Uniformly Distributed.*—Experiments on square slabs supported along the four edges indicate that the corners tend to curl up and that the dangerous section is approximately a diagonal section such as AC, Fig. 313. The bending moment, about the diagonal AC, of the forces that lie to one side of the diagonal may be found as follows: Let the load per unit area on the plate be w. The total load on the plate, then, is wb^2, the load W (Fig. 313) on one side of the diagram is $W = \frac{1}{2}wb^2$, and its action line passes through the centroid of the triangular area ACB; the centroid lies on the median line at a distance of $\frac{1}{3}h$ from AC. Now from the conditions of symmetry and of equilibrium, the reactions R_1 and R_2 are equal and each has a magnitude of $\frac{1}{4}wb^2$. The bending moment about the diagonal AC, then, is

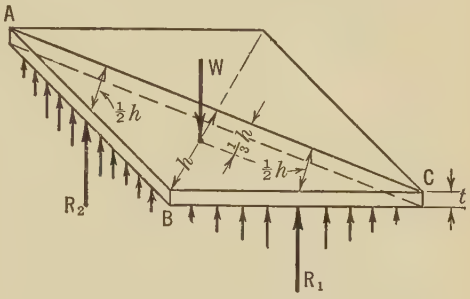


FIG. 313.—Forces acting on square flat plate.

$$M = (R_1 + R_2)\frac{h}{2} - W\frac{h}{3}$$

$$= \frac{wb^2h}{4} - \frac{wb^2h}{6} = \frac{1}{12}wb^2h$$

and the bending moment per unit width of the diagonal is

$$M_1 = \frac{1}{12}\frac{wb^2h}{AC} = \frac{1}{12}\frac{wb^2h}{2h}$$

$$= \tfrac{1}{24}wb^2 = 0.0417wb^2. \quad . \quad . \quad . \quad . \quad . \quad (260)$$

Now if the bending moment is assumed to be uniformly distributed

along the diagonal, the unit-stress s at any point on the surface of the plate at this diagonal section is

$$s = \frac{M_1 c}{I} = \frac{\frac{1}{24} wb^2 \times \frac{t}{2}}{\frac{1}{12} t^3};$$

whence

$$s = \frac{1}{4} \frac{wb^2}{t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (261)$$

The justification for assuming the moment (and stress) to be constant along the diagonal is given below.

Moment Coefficient.—If the plate were supported along two parallel sides only, it would act as a simple beam, the bending moment per unit width being $\frac{1}{8} wb^2$. The effect, then, of the supports along the additional two sides is to reduce the bending moment per unit width from $\frac{1}{8} wb^2$ to $\frac{1}{24} wb^2$. The numbers by which wb^2 is multiplied to obtain the bending moment per unit width are called moment coefficients and are commonly used to indicate the variation in bending moments (and hence of bending stresses) at various points along a given section of the plate.

The complete mathematical analyses, by several investigators, of the stresses in a homogeneous square flat plate supported along four edges and subjected to a uniformly distributed load are consistent in showing that the bending moment at the corners A and C, Fig. 313, is $0.0463 wb^2$, and that at the center of the diagonal it is $0.0369 wb^2$. That is, the moment coefficient along AC decreases from 0.0463 at the corners to 0.0369 at the center. The average of these is the value $\frac{1}{24} = 0.0417$ given by eq. (260).

This average value of $\frac{1}{24}$ (and eq. 261) has been used widely as a basis of design and may be justified by the fact that the high stresses at the corners are localized stresses which do not affect appreciably the behavior of the plate as a whole even though they exceed the proportional limit of the material, and that when the material begins to yield in a part of the diagonal section the stresses are redistributed so that they become more nearly uniform along the diagonal.

Square Plate Fixed at Edges.—If the plate is held rigidly (fixed) at the edges and subjected to a uniformly distributed load w per unit area, the maximum moment is the negative moment at the center of each of the edges of the plate. The moment coefficient for the bending moment at the center of the plate as found by mathematical analysis by several investigators is approximately 0.018; that is,

$$M_{\text{center}} = 0.018 wb^2 \text{ per unit width of plate} \quad . \quad . \quad . \quad (262)$$

and for the negative moment at the center of the edges the moment coefficient is approximately 0.050; that is,

$$M_{edge} = 0.050wb^2 \text{ per unit width of plate.} \quad . \quad . \quad (263)$$

Therefore the negative moment (and hence stress) at the edge is more than two and one-half times as large as that at the center. But since slight local yielding at the edges will redistribute the stress and give a more uniform distribution, the moment coefficient for use in design particularly with ductile material may be some value between those given in the above equations. If the average (0.034) of these values is used, the value of the stress is

$$s = \frac{Mc}{I} = \frac{0.034wb^2 \frac{t}{2}}{\frac{1}{12}t^3} = 0.20\frac{wb^2}{t^2} \text{ approximately.} \quad . \quad . \quad . \quad . \quad (264)$$

Nichols found, from experiments on square steel plates, a value of $s = 0.141\frac{wb^2}{t^2}$ corresponding to the first measurable permanent set in the plate, and Bach obtained from experiments a value of $s = 0.19\frac{wb^2}{t^2}$.

147. Stress in Rectangular Plate. *Supported along Four Edges; Load Uniformly Distributed.*—Experiments on rectangular slabs, mainly by Bach, indicate that the dangerous section is approximately a diagonal section such as AC, Fig. 314, the same as in square slabs. The pressure along the supporting edges probably varies about as shown in Fig. 314, since the corners are known to tend to curl up, but the resultant pressure on each edge acts at the mid-point of the edge.

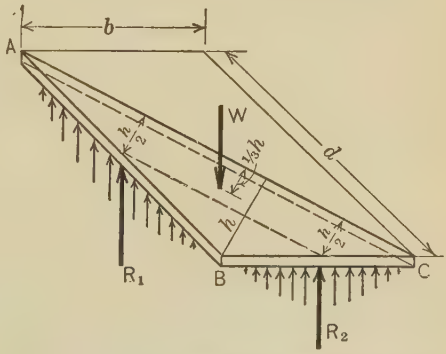


FIG. 314.—Forces acting on rectangular flat plate.

The average bending moment per unit width of diagonal may be found as was done with the square slab in the preceding article. Let the load per unit area on the plate be w. The total load on the plate then is $w \cdot ba$ and the load W (Fig. 314) on one side of the diagonal is $W = \frac{1}{2}wba$, and its action line passes through the centroid of the triangu-

lar area ACB; this centroid lies on the median line at a distance of $\frac{1}{3}h$ from AC. Now regardless of the values of R_1 and R_2 they have the same moment arm, and from the condition of symmetry and of equilibrium, their sum is equal to $\frac{wba}{2}$. The bending moment M about the diagonal AC, then, is

$$M = (R_1 + R_2)\tfrac{1}{2}h - W \cdot \tfrac{1}{3}h$$

$$= \tfrac{1}{2}wba \cdot \tfrac{1}{2}h - \tfrac{1}{2}wba \cdot \tfrac{1}{3}h = \tfrac{1}{12}wbah.$$

The bending moment per unit width of the diagonal is, therefore,

$$M_1 = \frac{\frac{1}{12}wbah}{AC}.$$

But

$$h = \frac{ab}{\sqrt{b^2 + a^2}} \quad \text{and} \quad AC = \sqrt{b^2 + a^2}.$$

Therefore,

$$M_1 = \frac{1}{12}\frac{a^2}{b^2 + a^2}wb^2 \qquad (265)$$

and thus the moment coefficient is $\frac{1}{12}\frac{a^2}{b^2 + a^2}$.

Now, if the bending moment were constant along the diagonal, the stress would also be constant and equal to

$$s = \frac{M_1 c}{I} = \frac{\frac{1}{12}\frac{a^2}{b^2 + a^2}wb^2\frac{t}{2}}{\frac{1}{12}t^3}$$

whence,

$$s = \frac{1}{2}\frac{a^2}{b^2 + a^2}\frac{wb^2}{t^2} \qquad (266)$$

in which s is the average bending unit-stress on the surface of the plate across the diagonal AC. If $a = b$, that is, if the plate is square, eqs. (265) and (266) reduce to eqs. (260) and (261), respectively. Eq. (266) is not applicable when the plate is very long and narrow, that is, when a is very large compared with b. For such a long narrow plate the supports along the short sides have little effect on the action in the plate and hence the plate acts approximately as would a simple beam having a span b; hence for such a plate the moment per unit width in the direction of the shorter dimension would be $\frac{1}{8}wb^2$.

The stress at the center of the plate is always greater in the direction

of the shorter span than in the direction of the larger span. This can be seen by considering two strips EF and GH (Fig. 315); the deflections of the two strips at the center are of course equal, but the shorter strip, being the stiffer, must carry the greater load, and hence the greater stress is developed in it; the moment per unit width causing stress in the direction of the shorter span decreases from the value $\frac{1}{8}wb^2$ as $\frac{b}{a}$ increases from a small value $\left(\frac{b}{a} = 0\right)$ and becomes $\frac{1}{24}wb^2$ when $a = b$ (see eq. (260)).

In Fig. 316 are given the bending moment per unit width across the diagonal at the corner (denoted by $M_{\text{diag.}}$), the bending moment per unit width at the center in the short span (denoted by M_{bc}) and the bending moment per unit width at the center in the long span (denoted by M_{ac}).

The curves and equations in Fig. 316 were obtained by Westergaard by making slight modifications in the results of rigid

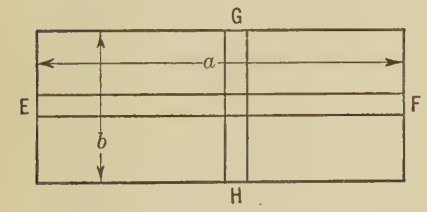


FIG. 315.

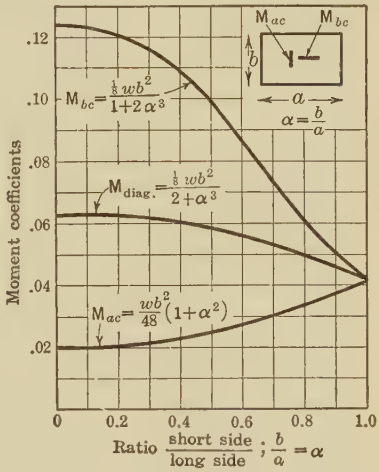


FIG. 316.—Bending moment per unit width in rectangular slabs with simply supported edges.

mathematical analyses. The modifications were made in order to obtain relatively simple expressions, and in doing so allowance was made for some redistribution of stress accompanying slight yielding at the portions of high (and more or less localized) stresses. It will be noted that the moment coefficient for a square slab $\left(\frac{b}{a} = 1\right)$ is $\frac{1}{24} = 0.0417$, and that for a long narrow slab $\left(\frac{b}{a} = 0\right)$ the moment coefficient for the short span is $\frac{1}{8} = 0.125$. For intermediate values of $\frac{b}{a}$, the

moment coefficient is always greater in the short span than elsewhere, and its value is intermediate between the limiting values of $\frac{1}{24}$ and $\frac{1}{8}$. But, as noted above, the average moment coefficient given in eq. (265) is probably, at least for ductile material, a better indication of the usable elastic strength of the plate than is the moment coefficient at the center of the short span.

Rectangular Plate with Fixed Edges.—If the plate is rigidly held (fixed) at the edges and subjected to a uniformly distributed load, the maximum moment per unit width occurs at the centers of the long edges, that is, at the fixed ends of the central strip of the short span.

Two limiting cases of a fixed-edged rectangular slab will be considered first. If the plate is very long and narrow $\left(\frac{b}{a} = 0\right)$, the forces at the short ends of the plate will have negligible effect on the moment in the central part of the plate, and hence the plate may be considered to be a fixed-ended beam with a span equal to the short dimension of the plate; therefore, the negative moment per unit width, M_{be}, at the fixed edges of the short span is $\frac{1}{12}wb^2$, and the positive moment, M_{bc}, at the center of the short span is $\frac{1}{24}wb^2$. The other limiting case is that of the square slab $\left(\frac{b}{a} = 1\right)$ for which the moment coefficient at the center of the edges, as stated in eq. (263), is approximately 0.05 and the moment coefficient at the center is 0.018.

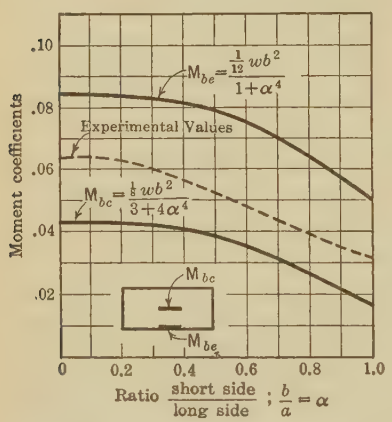


Fig. 317.—Bending moment per unit width in rectangular slabs with fixed edges.

For plates having other values of $\frac{a}{b}$, the maximum negative moment M_{be} and the maximum positive moment M_{bc} are given in Fig. 317. These values were obtained by Westergaard by simplifying the results obtained by mathematical analysis. As explained in the preceding articles, owing to the advantageous redistribution of stresses accompanying slight yielding of the plate at points of maximum stress, and the lateral restraint caused by stresses at right angles to each other, the plate is somewhat stronger than is indicated by the results of mathematical analysis.

For plates made of ductile material, the maximum moment used

in design should probably be about the average of the values of M_{be} and M_{bc} given in Fig. 317. Bach, from the results of experiments, recommends the moment coefficients given by the dotted line in Fig. 91. Montgomerie experimented on steel plates 24 in. by 48 in. $\left(\frac{b}{a} = 0.5\right)$ with thicknesses varying from $\frac{1}{8}$ in. to $\frac{3}{4}$ in. and found the maximum moment per unit width to be approximately $0.042wb^2$. His results indicate that there is not much difference in the value of the stress at the center and at the end of the short span.

ILLUSTRATIVE PROBLEM

Problem 446. A water tank 12 ft. deep and 9 ft. square is to be made of structural steel plate. The sides of the tank are divided into nine panels by two vertical supports or stiffeners and two horizontal supports; that is, each panel is 3 ft. wide and 4 ft. high and the average head of water on a lower panel is 10 ft. Determine the thickness of the plate for the lower panels, using a working stress of 14,000 lb. per sq. in.

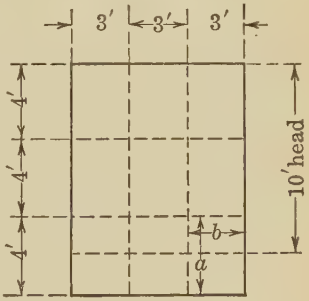


FIG. 318.

Solution.—The mean pressure on a bottom panel (Fig. 318) is $w = \dfrac{10 \times 62.4}{144} = 4.33$ lb. per sq. in., and this pressure will be assumed to be uniform over the panel.

(*a*) If the panel is assumed to be supported around the edges, then the average moment coefficient for a diagonal section is, according to eq. (265).

$$\frac{1}{12}\frac{a^2}{b^2 + a^2} = \frac{1}{12}\frac{16}{9 + 16} = 0.0534.$$

From Fig. 316 it will be noted that this coefficient is considerably less than that at the center of the short span, but, as stated in Art. 147, this value is probably the more significant. Accordingly the significant stress, across a diagonal is (see eq. 266)

$$s = \frac{1}{2}\frac{a^2}{a^2 + b^2}\frac{wb^2}{t^2} \quad \text{or} \quad t = ab\sqrt{\frac{w}{2(a^2 + b^2)s_w}}$$

whence

$$t = 48 \times 36\sqrt{\frac{4.33}{2(3600)14{,}000}} = 0.359 \text{ in.}$$

(*b*) If the panel is assumed to be fixed at the edges, then from Fig. 317 we have, approximately, for $\frac{b}{a} = \frac{3}{4} = 0.75$,

$$M = 0.042wb^2 = 0.042 \times 4.33 \times (36)^2 = 236 \text{ lb.-in.}$$

and

$$s = \frac{M}{I/c} = \frac{6M}{t^2}, \quad \text{whence} \quad t = \sqrt{\frac{6M}{s}} = \sqrt{\frac{6 \times 236}{14{,}000}} = 0.318 \text{ in.}$$

The plate is probably restrained somewhat at the edges but not fixed, and a value of t between 0.318 in. and 0.359 in. would be adequate for strength, but a commercial size and other considerations, such as rusting, etc., might dictate a larger thickness.

PROBLEMS

447. The cylinder of a steam engine is 16 in. in diameter and the maximum steam pressure is 100 lb. per sq. in. Find the thickness of a flat plate cylinder head of steel if the working stress is 12,000 lb. per sq. in. (*a*) Assuming that the edges are simply supported, (*b*) assuming the edges are fixed.

448. A structural steel trap door is 6 ft. long, 3 ft. wide and $\frac{3}{8}$ in. thick. How large a uniform load will it support without being given a permanent deformation? Assume elastic limit = 35,000 lb. per sq. in. Assume that the edges are simply supported. *Ans.* 9.5 lb./in.^{2}

449. A rectangular water tank is 12 ft. deep and 12 ft. wide. It is divided into two compartments by a vertical steel flat plate. This flat plate is divided in four square panels by a vertical and a horizontal stiffener in addition to the stiffener across the top. What thickness should the plate have, assuming a working stress of 16,000 lb. per sq. in.? One compartment may be full and one empty. Assume the edges of the plate are fixed.

450. A cast-iron flat valve gate 12 in. in diameter is subjected to a water pressure equivalent to a head of 200 ft. Find the thickness of the gate, using a working stress of 2000 lb. per sq. in. Assume that the edges are simply supported.

Ans. $t = 1.25$ in.

CHAPTER XVIII

THICK-WALLED CYLINDERS

148. The Problem Defined.—By the term thin-walled or thin-shelled cylinder is meant a cylinder in which the circumferential stress (frequently called hoop-tension stress) in the shell may, without introducing appreciable errors in the analysis, be assumed to be constant throughout the thickness of the shell when the cylinder is subjected to an internal fluid pressure.

Most boilers, tanks, steam and water pipes, etc., may be treated satisfactorily as thin-shelled circular cylinders. If, however, the thickness of the shell or wall is relatively large, as in the case of guns, pipe to hydraulic presses, and similar high-pressure hydraulic and steam machinery, the variation in the stress from the inner surface to the outer surface is relatively large, and the value of the stress found from the ordinary formula for a thin-shelled cylinder may not be a satisfactory measure of the significant stress in the cylinder.

The problem here considered is to find the circumferential stress s_t and the radial stress s_r at any point in the material of a thick-walled cylinder in terms of the internal and external pressures, and radii of the internal and external surfaces, and the distance from the central axis to the point at which the stress is desired.

149. Lamé's Solution.—Two equations involving the two unknown stresses (s_t and s_r) may be found by making use of the assumptions; (*a*) that the thick cylinder is composed of concentric thin-shelled cylinders to each of which may be applied the method used for determining stresses in a thin-shelled cylinder; and (*b*) that the strains of all longitudinal fibers of the cylinder are equal, that is, a plane transverse section remains plane and parallel to itself; this assumption would not be reliable for sections near the ends of a closed cylinder where the strains of the fibers are influenced by the bending of the head or end of the cylinder, particularly if the cylinder end is flat.

Let Fig. 319 represent a thick-walled cylinder that has closed ends and is subjected to internal and external fluid pressures of intensities p_1 and p_2, respectively. Consider the forces acting on one-half of one of the imaginary concentric shells of which the thick cylinder is assumed

to be composed; the imaginary shell has a radius x and a thickness dx; the intensity of stress on the inner surface of the shell is s_r and that on the outer surface is $(s_r + ds_r)$, where ds_r is the increment in s_r in the distance dx due to the variation in the intensity of pressure from p_1 at the inner surface of the cylinder to p_2 at the outer surface. Since the half-shell is in equilibrium, the sum of the components, in the direction of s_t, of all forces acting on the half-shell must be equal to zero; or, the resultant of the pressures on the inner and outer surfaces of the half-shell must be equal to the total circumferential stress on a diametral plane. For convenience, the length of the shell is assumed to be unity; further, both

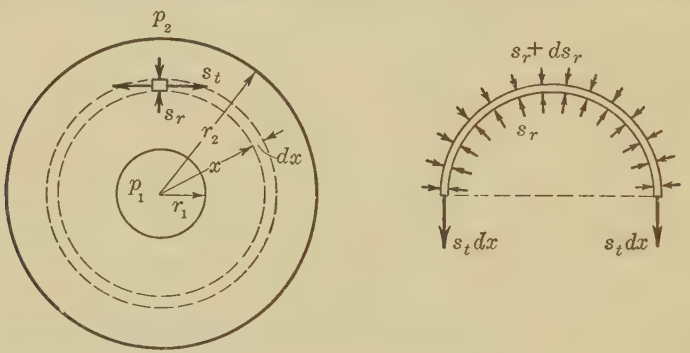


FIG. 319.—Stresses in thick-walled cylinder.

s_t and s_r are considered to be positive. Referring to Fig. 319, we obtain from equilibrium,

$$s_r \cdot 2x - (s_r + ds_r) \cdot 2(x + dx) = 2s_t dx.$$

By neglecting the term $dx ds_r$, which is very small compared to the other quantities involved, we obtain

$$s_t = -s_r - x\frac{ds_r}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (267)$$

This equation gives one relation between s_t and s_r.

Another relation is found by using the assumption that the longitudinal strains of all fibers are equal. Let the longitudinal stress in the fibers due to the pressure against the ends of the cylinder be s_0. Then (see Fig. 324),

$$\text{Longitudinal strain due to } s_0 = \frac{s_0}{E},$$

$$\text{Longitudinal strain due to } s_t = -m\frac{s_t}{E},$$

$$\text{Longitudinal strain due to } s_r = m\frac{s_r}{E},$$

where m is Poisson's ratio. The total longitudinal unit-strain ε is

$$\varepsilon = \frac{s_0}{E} + m\frac{s_r}{E} - m\frac{s_t}{E}. \quad \ldots\ldots \quad (268)$$

But ε and s_0 are assumed to be constant for all fibers, and m and E are assumed to be constant for the material, and hence from eq. (268) we obtain

$$s_t - s_r = \text{constant} = 2a \text{ (say)}, \quad \ldots\ldots \quad (269)$$

which gives the second relation between s_t and s_r. Now from eqs. (267) and (269) we have,

$$2a = -2s_r - \frac{x\,ds_r}{dx}. \quad \ldots\ldots \quad (270)$$

But the right-hand member of this equation when multiplied by x becomes the derivative, with respect to x, of $-(x^2 s_r)$, and hence we may write

$$\frac{d(x^2 s_r)}{dx} = -2ax.$$

By integrating this equation we obtain

$$x^2 s_r = -ax^2 + b, \quad \ldots\ldots \quad (271)$$

where b is a constant of integration. Therefore,

$$s_r = \frac{b}{x^2} - a \quad \ldots\ldots \quad (272)$$

and from eq. (269)

$$s_t = \frac{b}{x^2} + a. \quad \ldots\ldots \quad (273)$$

The values of the constants a and b are found by substituting in eq. (272) known pairs of values of s_r and x as obtained from the dimensions and the physical conditions under which the cylinder is used. For example, when the cylinder is subjected to pressures on both internal and external surfaces we have

$$s_r = p_1 \text{ when } x = r_1, \quad \text{and} \quad s_r = p_2 \text{ when } x = r_2$$

and hence from eq. (272) we obtain

$$p_1 = -a + \frac{b}{r_1^2} \quad \text{and} \quad p_2 = -a + \frac{b}{r_2^2}, \quad \ldots\ldots \quad (274)$$

from which a and b are found to be

$$a = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} \quad \text{and} \quad b = \frac{r_1^2 r_2^2}{r_2^2 - r_1^2}(p_1 - p_2). \quad . \quad (275)$$

These values of a and b may now be substituted in eqs. (272) and (273) to obtain the expression for s_r and s_t at any point distant x from the center of the cylinder. The expressions are:

$$s_t = \frac{p_1 r_1^2 - p_2 r_2^2 + \dfrac{r_1^2 r_2^2}{x^2}(p_1 - p_2)}{r_2^2 - r_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (276)$$

$$s_r = \frac{p_2 r_2^2 - p_1 r_1^2 + \dfrac{r_1^2 r_2^2}{x^2}(p_1 - p_2)}{r_2^2 - r_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (277)$$

It is evident from eqs. (273) and (275) or from (276) that the maximum value of s_t occurs at the inner surface where x has its minimum value, r_1 (see Fig. 322 and Prob. 452). The maximum value of s_r will always be the larger of the two pressures p_1 and p_2.

Signs.—Both s_t and s_r were considered to be positive in the above derivation, and hence when the expression for s_t gives a positive value, s_t is a tensile stress, and when it gives a negative value, s_t is a compressive stress; whereas when the expression for s_r gives a positive value, s_r is a compressive stress, and it will always be positive.

Maximum Stresses with Both Internal and External Pressures.—By making x in the above equations equal to r_1, the maximum stresses (at the inner surface) are found to be

$$\text{Max } s_t = \frac{p_1(r_1^2 + r_2^2) - 2p_2 r_2^2}{r_2^2 - r_1^2}, \quad . \quad . \quad . \quad . \quad . \quad (278)$$

and

$$\text{Max } s_r = p_1 \text{ if } p_1 > p_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (279)$$

Special Cases: Maximum Stresses

Internal Pressure Only.—If the internal pressure is p_1 and the external pressure is zero ($p_2 = 0$), as in most hydraulic pipes, tanks, etc., eqs. (276) and (277) reduce to

$$s_t = p_1 \frac{r_1^2}{r_2^2 - r_1^2}\left(\frac{r_2^2}{x^2} + 1\right), \quad . \quad . \quad . \quad . \quad (280)$$

$$s_r = p_1 \frac{r_1^2}{r_2^2 - r_1^2}\left(\frac{r_2^2}{x^2} - 1\right). \quad . \quad . \quad . \quad . \quad (281)$$

These equations show that the maximum values of s_t and s_r occur at the inner surface, that is, when x has its least value, r_1; they also show that, for any value of x, s_t is always greater than s_r. The maximum value of s_t as found from eq. (280) by making $x = r_1$ is

$$\text{Max } s_t = p_1 \frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (282)$$

and the maximum value of s_r for this case is p_1; these maximum stresses are indicated in Fig. 320. The maximum shearing stresses in the cylinder is considered in the next article.

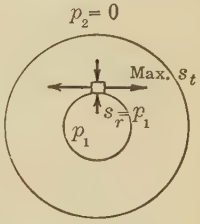


FIG. 320.

It is important to note that eq. (282) shows that s_t is always greater than p_1, and hence p_1 must always be less than the proportional limit of the material; therefore, when large internal pressures are required, as in guns, hoops are shrunk on the cylinder to cause initial compressive stresses, or an underbored gun is strained beyond the proportional limit by hydraulic pressure, so that, on relief of pressure, the metal of the bore is in a state of initial compression, the outer layers being in moderate initial tension.

In the following table is given the ratio of the stress obtained from the thick-cylinder formula (282) to that obtained by the thin-cylinder formula $\left(s_{\text{thin}} = \frac{pr}{t}\right)$ for various values of $\frac{r_2}{r_1}$.

$\frac{r_2}{r_1} =$	1	1.2	1.4	1.6	1.8	2.0	2.5
$\frac{s_{\text{thick}}}{s_{\text{thin}}} =$	1	1.10	1.23	1.37	1.51	1.67	2.07

For example, the maximum circumferential stress in a cylinder in which $r_2 = 1.4r_1$ is 1.23 times the value of s obtained from the thin-cylinder formula. (Fig. 322 shows the variation of circumferential stress across a section for which $r_2 = 2r_1$.)

External Pressure Only.—If the external pressure is p_2 and the internal pressure is zero ($p_1 = 0$), eqs. (276) and (277) reduce to

$$s_t = -p_2 \frac{r_2^2}{r_2^2 - r_1^2}\left(1 + \frac{r_1^2}{x^2}\right), \quad . \quad . \quad . \quad . \quad (283)$$

$$s_r = p_2 \frac{r_2^2}{r_2^2 - r_1^2}\left(1 - \frac{r_1^2}{x^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (284)$$

These equations show that the maximum value of s_t occurs at the inner surface (when $x = r_1$), and that it is a compressive (negative) stress; its value is

$$s_t = -2p_2 \frac{r_2^2}{r_2^2 - r_1^2}. \quad \ldots\ldots\ldots \quad (285)$$

The maximum value of s_r is, of course, at the outer surface, and its value is p_2; as indicated in Fig. 321, the maximum values of s_t and s_r do not occur at the same point in the cylinder for this case.

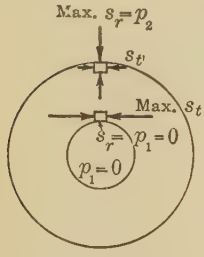


FIG. 321.

ILLUSTRATIVE PROBLEM

Problem 451.—The cylinder of a field gun has an internal diameter of 4.70 in. and an external diameter of 7.72 in. The tensile and compressive proportional limit of the material is 50,000 lb. per sq. in. (*a*) Find the maximum internal pressure to which the cylinder may be subjected without stressing the material above the proportional limit. (*b*) Find the maximum external pressure that may be applied (by shrinking on hoops) without causing a stress greater than the proportional limit.

Solution.—(*a*) By use of eq. (282) we have

$$\text{Max } s_t = p_1 \frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}$$

$$50{,}000 = p_1 \frac{(3.86)^2 + (2.35)^2}{(3.86)^2 - (2.35)^2} = \frac{20.42}{9.38} p_1$$

$$p_1 = \frac{50{,}000}{2.18} = 23{,}000 \text{ lb. per sq. in.}$$

Hence, the maximum internal pressure that may be applied when the external pressure is zero is 23,000 lb. per sq. in.

(*b*) By use of eq. (285) we have

$$\text{Max } s_t = -2p_2 \frac{r_2^2}{r_2^2 - r_1^2}$$

$$-50{,}000 = -2p_2 \frac{(3.86)^2}{(3.86)^2 - (2.35)^2}$$

$$p_2 = \frac{50{,}000}{3.18} = 15{,}700 \text{ lb. per sq. in.}$$

Hence, the maximum external pressure that may be applied when the internal pressure is zero is 15,700 lb. per sq. in.

PROBLEMS

452. Plot a curve (similar to AB in Fig. 322) the ordinates to which show the values of the circumferential unit-stress at various distances from the center of the cylinder. (*a*) Let $r_1 = 1$ in., $r_2/r_1 = 1.2$, $p_1 = 1000$ lb. per sq. in., and $p_2 = 0$. (*b*) Plot a similar curve using the value $r_2/r_1 = 4$, the other data remaining the same. In obtaining the curve in Fig. 322 the value used for r_2/r_1 was 2.

453. A cylinder with an internal diameter of 8 in. and an external diameter of 24 in. is subjected to an internal pressure, only, of 10,000 lb. per sq. in. Find the maximum circumferential and radial unit-stresses and state where they occur in the cylinder.

Ans. $s_t = 12{,}500$ lb. per sq. in.
$s_r = 10{,}000$ lb. per sq. in.

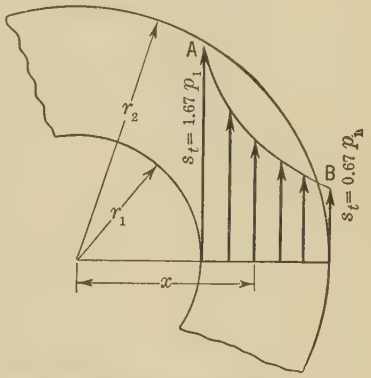


FIG. 322.—Distribution of circumferential stress in thick-walled cylinder.

454. If the cylinder in Prob. 453 is subjected to an external pressure of 10,000 lb. per sq. in. and no internal pressure, find the maximum circumferential and radial unit-stresses, and state where in the cylinder they occur. *Ans.* $s_t = -22{,}500$ lb. per sq. in.

455. A steel air tank having an internal diameter of 5 ft. is required to resist an internal pressure of 5000 lb. per sq. in. The tensile and compressive proportional limit of the material is 30,000 lb. per sq. in. What should be the thickness of the wall if the working value of the circumferential stress is two-thirds of the proportional limit (that is, a so-called factor of safety of 1.5 based on the proportional limit is used).

Ans. $t = 8.70$ in.

456. A hoop shrunk on a hollow cylinder exerts an external pressure of 3000 lb. per sq. in. on the cylinder; an internal pressure of 10,000 lb. per sq. in. is then applied to the cylinder. The internal diameter of the cylinder is 3 in. and the external diameter is 5 in. The external diameter of the hoop is 8 in. What is the maximum circumferential stress in the cylinder (*a*) before the internal pressure is applied? (*b*) after the internal pressure is applied? (*c*) Considering the 3000 lb. per sq. in. pressure as the only pressure acting on the inside of the hoop, what is the maximum stress in the hoop? It is assumed that the proportional limit of the material is greater than the calculated stresses.

150. Maximum Shearing Stress.—According to the maximum shearing stress theory, the breakdown of elastic action in a material subjected to any combination of stress begins only when the maximum shearing stress reaches the shearing proportional limit of the material.

As shown in Art. 84 of Chapter IX, the maximum shearing stress at any point is equal to one-half the algebraic difference of the maximum and minimum principal stresses at the point. Since s_0 is usually small

compared to s_t, and since s_r is a negative tensile stress, the maximum principal stress is s_t and the minimum is $-s_r$. Hence

$$\text{Max } s_s = \tfrac{1}{2}[s_t - (-s_r)] = \tfrac{1}{2}(s_t + s_r).$$

By introducing the values for s_t and s_r as given by eqs. (276) and (277), we obtain the expression for the maximum shearing stress s_s at any point at the distance x from the center of the cylinder:

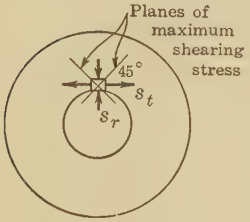


FIG. 323.

$$s_s = \frac{r_1^2 r_2^2}{x^2} \frac{(p_1 - p_2)}{(r_2^2 - r_1^2)}. \quad . \quad . \quad (286)$$

This stress occurs on a plane making an angle of 45° with the directions of s_t and s_r as indicated in Fig. 323. The maximum value of s_s occurs at the inner surface, that is, where $x = r_1$, and hence may be obtained easily from eq. (286).

Internal Pressure Only.—If the tube is subjected to internal pressure only, the maximum shearing stress is found by making $x = r_1$ and $p_2 = 0$, in eq. (286), to be

$$\text{Max } s_s = p_1 \frac{r_2^2}{r_2^2 - r_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (287)$$

The maximum shearing unit-stress for other combinations of p_1 and p_2 may be found in a similar way.

PROBLEMS

457. The shearing proportional limit of the material in the gun described in Prob. 451 is 0.6 of the tensile proportional limit. If the maximum allowable shearing stress in the gun is specified to be three-quarters of the shearing proportional limit of the material, find the maximum allowable internal pressure, assuming the external pressure to be zero.

458. Find the maximum shearing unit-stress in the cylinder described in Prob. 453, and state where this stress is located in the tube and the direction of the plane on which the stress acts. *Ans.* $s_s = 11{,}250$ lb. per sq. in.

459. The shearing proportional limit of the material of the tank described in Prob. 455 is 0.6 of the tensile proportional limit. If the maximum allowable value of the shearing stress is two-thirds of the shearing proportional limit, what should be the thickness of the tank wall? *Ans.* $t = 9.3$ in.

151. Maximum Strain and Maximum Value of $E\varepsilon$.—According to the maximum principal strain theory (see Art. 89), the breakdown of elastic action in a material subjected to any combination of stresses begins only when the maximum principal *strain* reaches a given value,

the given value being that which occurs when inelastic action begins in a bar subjected to a single direct tensile stress; this limiting strain, ε_e, is the strain that occurs simultaneously with the tensile proportional limit and is, of course, equal to $\frac{s_e}{E}$ where s_e is the proportional limit.

It is important, therefore, to obtain expressions for the maximum strain (ε_{max}) in a thick cylinder, because the value of $E\varepsilon_{max}$, according to the maximum strain theory, must not be allowed to exceed the value s_e if the breakdown of elastic action is to be prevented.

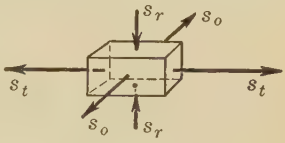


FIG. 324.

The most severe straining action occurs at the inner fibers where the circumferential stress is a maximum; and the direction of the maximum strain will be the same as that of the circumferential stress s_t (Fig. 324). The value of the maximum strain is

$$\varepsilon_{max} = \frac{s_t}{E} + m\frac{s_r}{E} - m\frac{s_0}{E}. \quad . \quad . \quad . \quad . \quad . \quad (288)$$

By substituting, in this equation, the values of s_t and s_r found in Art. 149 for any given combination of p_1 and p_2 (eqs. 276 and 277), we find the maximum value of the strain (ε_{max}) or of $E\varepsilon_{max}$.

Internal Pressure Only.—Let it be assumed that the longitudinal stress s_0 and the external pressure p_2 are zero. Then from eqs. (282) and (288) we have

$$E\varepsilon_{max} = p_1\left[\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} + m\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (289)$$

For steel m is approximately $\frac{1}{4}$; hence for a steel cylinder,

$$E\varepsilon_{max} = p_1\left[\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} + \frac{1}{4}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (290)$$

and, as noted above, the allowable value of $E\varepsilon_{max}$ must be less than the proportional limit of the material.

PROBLEMS

460. The tube described in Prob. 453 is made of steel. The maximum allowable value of the strain is specified to be two-thirds of that corresponding to the tensile proportional limit of the material. The value of the proportional limit is 40,000 lb. per sq. in. Find the maximum allowable internal pressure, assuming the external pressure to be zero. Use $m = 0.25$. *Ans.* $p_1 = 17{,}780$ lb. per sq. in.

461. The maximum allowable strain in the gun described in Prob. 451 is that corresponding to the proportional limit (or $E\varepsilon_{max} = s_e = 50{,}000$). If the external pressure is 15,700 lb. per sq. in., find the maximum allowable internal pressure.

Ans. 41,100 lb. per sq. in.

462. The maximum allowable value of the strain in the gun described in Prob. 451 is three-quarters of that corresponding to the proportional limit $\left(\varepsilon_{max} = \frac{3}{4}\frac{s_e}{E}\right)$. Find the maximum allowable internal pressure to which the gun may be subjected, assuming the external pressure to be zero.

CHAPTER XIX

TORSIONAL RESISTANCE OF BARS HAVING NON-CIRCULAR CROSS-SECTIONS

152. Introduction.—If a bar is twisted by couples that lie in planes perpendicular to the axis of the bar, any transverse section of the bar, between the planes of the twisting couples is rotated relative to adjacent sections, and hence torsional shearing strains and stresses in the section are developed.[1] Further, at any point in a body a shearing stress on one plane requires that a shearing stress of equal intensity act on a plane at right angles to the first plane (Art. 27, Chapter III), and hence, at any point in the bar subjected to torsion, shearing stresses of equal intensity occur on the transverse and longitudinal planes through the point as indicated in Fig. 325.

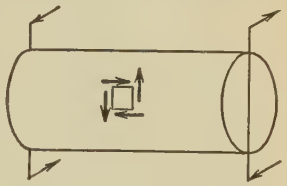


FIG. 325.

Circular Section.—Now if the bar has a *constant circular cross-section,* all points in any transverse plane turn through equal angles, relative to any other transverse plane; and a plane section before twisting is still a plane section after twisting. From this it follows that when a bar with a circular cross-section is twisted, the shearing strain on any transverse section increases directly with the distance from the axis of the shaft. Further, within the proportional limit, the shearing unit-stress is proportional to the shearing unit-strain, and hence the shearing unit-stress at any point on the transverse section of a cylindrical bar varies directly as the distance of the point from the center of the shaft.

The torsional moment of the shearing stresses on any transverse section of the bar (called the resisting moment) holds in equilibrium the moment of the external forces that lie to one side of the section; this moment is called the external moment or twisting moment and is denoted by T. Hence we have

$$\text{Twisting moment} = \text{resisting moment.}$$

[1] In this chapter it is assumed that all cross-sections of the bar are free to warp.

And since the shearing unit-stress on the section varies directly with the distance from the axis of the shaft, the expression for the resisting moment as found in Art. 21 is $\frac{s_s J}{c}$. Therefore,

$$T = \frac{s_s J}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (291)$$

in which s_s is the shearing unit-stress at any point in the section, the distance of the point from the axis of the shaft being c; and J is the polar moment of inertia of the circular section $\left(J = \frac{\pi d^4}{32}\right)$.

The angle of twist, θ, of a cylindrical shaft when subjected to a twisting moment T is (see Art. 24)

$$\theta = \frac{Tl}{JE_s} \quad \text{or} \quad \phi = \frac{\theta}{l} = \frac{T}{JE_s} \quad . \quad . \quad . \quad . \quad . \quad (292)$$

in which l is the length of the shaft, E_s is the shearing modulus of elasticity of the material, and ϕ is the angle of twist per unit of length. Eqs. (291) and (292) apply only to shafts having constant circular cross-sections and are valid only when the shearing stress in the shaft does not exceed the shearing proportional limit of the material.

Non-circular Section.—In 1853, Saint Venant, in his classical memoir on the torsion of prisms, showed that if a bar whose transverse cross-section is not circular is twisted by applying moments at its ends, a plane transverse section before twisting does *not* remain a plane section after twisting; it becomes a warped surface, and this warping is accompanied by an increase of shearing stress in some parts of the shaft and a decrease in other parts as compared with stresses that would occur if the section did not warp but remained a plane as in a bar having a circular cross-section.

For example, in a bar with an elliptical section the shearing stress on any transverse section has a maximum value at the end of the minor axis, that is, at the point on the surface *nearest* to the axis of the shaft; whereas, if the plane section remained plane so that the shearing strain (and also stress) at any point were proportional to the distance of the point from the axis of twist, the strain and stress on the surface of the bar would be a minimum at the end of the minor axis.

Similarly, the maximum shearing stress in a bar with a rectangular section is at the center of the long side, that is, at a point on the surface *nearest* to the axis of the bar; and the shearing stress at each corner of the section is zero.

If a section has a sharp internal corner (re-entrant angle) such as

that caused by a keyway in a shaft, the warping of a transverse section causes at the internal corner a shearing stress which theoretically is infinitely large.

As a rule, structural members such as I-beams, channels, etc., are used to resist bending loads, but frequently these members are subjected to torsion in addition to bending, and although the torsional moment may be relatively small the lack of torsional strength and stiffness of such members makes their torsional behavior of importance.

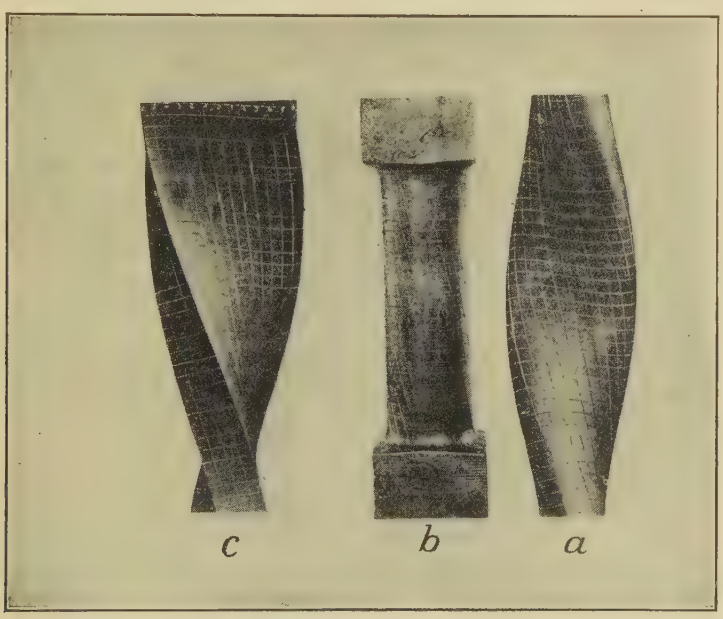


FIG. 326.—Views showing warping of sections of bars having non-circular cross-section. (From C. Bach.)

The analysis given in the next article of the torsional stresses in a bar having a rectangular section is an approximate but very useful and relatively simple solution due mainly to C. Bach (see Elasticität und Festigheit, 8th Ed.).

153. Torsion of Bar Having a Rectangular Cross-section.—When a bar having a rectangular section is twisted it takes the form shown in Fig. 326(*c*). The warping of the sections causes the shearing strain (and hence shearing stress) to be maximum at the center of the long side (at *A*, Fig. 327) and to be zero at the corners (at *C*). These facts are obtained from a study of Fig. 326(*c*), which shows that the greatest distortion of the squares occurs at the middle of the long

side and the least at the edge of the bar. The fact that the stress must be zero at the corner may be shown as follows: The stress at any point in the surface must be tangent to the surface; therefore, since C is a point on the side AC (Fig. 327) and also on the side CB, the stress at C must be zero since it could not act tangent to both sides.

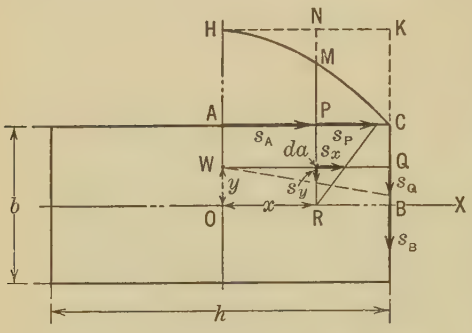


FIG. 327.—Torsion of rectangular section.

Fig. 326(*c*) indicates that the strain (and hence stress) varies from A to C (Fig. 327) approximately as the ordinates of a parabola. Thus in Fig. 327 the stresses along AC are indicated by the ordinates to the parabola, HMC having its vertex at H. In Fig. 327, let s_A represent the stress at A and s_P represent the stress at any point P on the periphery between A and the corner C. Now, if HMC is a parabola in which AH represents s_A and $\overline{PM}$ represents s_P, we have from the equation of a parabola

$$\frac{\overline{MN}}{\overline{CK}} = \frac{x^2}{\left(\frac{h}{2}\right)^2}; \text{ whence } \frac{(s_A - s_P)}{s_A} = \left(\frac{2x}{h}\right)^2$$

or

$$s_P = s_A\left[1 - \left(\frac{2x}{h}\right)^2\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (293)$$

Similarly,

$$s_Q = s_B\left[1 - \left(\frac{2y}{b}\right)^2\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (294)$$

Now it will be assumed that the stress component s_x (Fig. 327) along any line (such as WQ) parallel to AC varies in the same way as does the stress along AC, but that the value of s_x decreases directly from the value s_P on the surface at P to zero at R on the major axis; a similar assumption is made with respect to s_y. That is, if s_x and s_y denote the components of the stress at any point in the bar

$$s_x = \frac{y}{\frac{1}{2}b}\, s_P = \frac{2y}{b}\, s_A\left[1 - \left(\frac{2x}{h}\right)^2\right] \quad . \quad . \quad . \quad . \quad (295)$$

and

$$s_y = \frac{x}{\frac{1}{2}h}\, s_Q = \frac{2x}{h}\, s_B\left[1 - \left(\frac{2y}{b}\right)^2\right]. \quad . \quad . \quad . \quad . \quad (296)$$

Further, it is assumed that s_A and s_B are inversely proportional to the distances of A and B from the center of the bar. That is,

$$s_B = \frac{b}{h} s_A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (297)$$

These assumptions are in accord with the general behavior of the bar as indicated by Fig. 326(c), although they must be regarded only as approximate since the exact manner of variation of stress cannot be determined in this way.

Now the external twisting moment T is equal to the internal resisting moment, and hence we may write

$$T = \int (s_x da\, y + s_y da\, x). \quad . \quad . \quad . \quad . \quad . \quad (298)$$

By use of eqs. (295), (296) and (297), the right-hand member of this equation may be expressed in terms of the maximum unit-stress s_A and the dimensions of the area. Thus,

$$T = \frac{2s_A}{b}\left[\int y^2 da - \frac{4}{h^2}\int x^2 y^2 da\right] + \frac{2b}{h^2} s_A\left[\int x^2 da - \frac{4}{b^2}\int x^2 y^2 da\right]. \quad (299)$$

But

$$\int y^2 da = I_x = \tfrac{1}{12} h b^3; \quad \int x^2 da = I_y = \tfrac{1}{12} b h^3, \text{ and } \int x^2 y^2 da = \frac{b^3 h^3}{144}.$$

Hence eq. (299) becomes

$$T = \tfrac{2}{9} b^2 h s_A \quad \text{or} \quad s_A = \frac{9}{2}\frac{T}{b^2 h}, \quad . \quad . \quad . \quad . \quad (300)$$

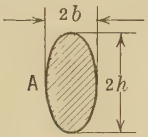
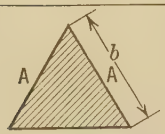
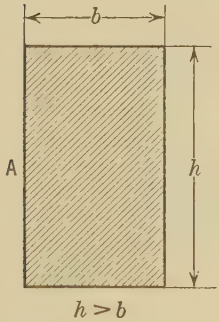
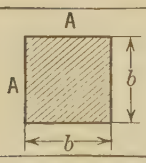
in which s_A is the maximum shearing unit-stress in the bar when subjected to a twisting moment T, provided that b is not greater than h and that the stress does not exceed the shearing proportional limit of the material. If T is expressed in pound-inches and b and h in inches, s_A will be expressed in pounds per square inch.

The above expression for the maximum stress must not be expected to give reliable results for a very thin section, that is, when h is very large compared to b. When h has a value of $1b$ to $3b$, the value of s_A, though only approximate, may be considered to be reliable. If eq. (300) is written

$$T = \alpha b^2 h s_A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (301)$$

TABLE 10

APPROXIMATE FORMULAS FOR TORSIONAL SHEARING STRESS AND ANGLE OF TWIST, OBTAINED FROM MATHEMATICAL ANALYSIS

Cross-section	Relation between Shearing Unit-stress and Twisting Moment	Relation between Angle of Twist and Twisting Moment
	$s_s = \frac{2T}{ab}$ $= \frac{2T}{\pi h b^2}$	$\phi = 4\pi^2 \frac{J}{a^4} \cdot \frac{T}{E_s}$ $= \frac{h^2+b^2}{\pi h^3 b^3} \frac{T}{E_s}$
 Equilateral Triangle	$s_s = \frac{20T}{b^3}$	$\phi = \frac{80}{b^4\sqrt{3}} \frac{T}{E_s} = \frac{46.2}{b^4} \cdot \frac{T}{E_s}$
 $h > b$	$s_s = \frac{8\beta T}{\mu h b^2}$	$\phi = \frac{16}{\mu h b^3} \frac{T}{E_s}$
	Saint Venant gives the following approximate formulas for β and μ which yield values that agree within 4 per cent of the exact values: $\beta = \frac{3}{8}\left(1+0.6\frac{b}{h}\right)\mu$ $\mu = \frac{16}{3} - 3.36\frac{b}{h}\left(1-\frac{1}{12}\frac{b^4}{h^4}\right)$ Using these values we find:	
	$s_s = \left(\frac{15h+9b}{5h^2b^2}\right) T$	$\phi = \frac{16}{\left[\frac{16}{3} - 3.36\frac{b}{h}\left(1-\frac{1}{12}\frac{b^4}{h^4}\right)\right] h b^3} \cdot \frac{T}{E_s}$ $= \frac{4\pi^2 J}{a^4} \cdot \frac{T}{E_s}$ approximate. (Error is small if $h > 2b$)
	For a square the above formulas reduce to: $s_s = \frac{24}{5} \frac{T}{b^3}$	$\phi = \frac{7.11}{b^4} \cdot \frac{T}{E_s}$
Any compact section *not* containing re-entrant angles.	$s_s = \frac{4\pi^2 J}{a^4 y} \cdot T$ y = distance of most remote edge from center of bar	$\phi = k \frac{J}{a^4} \cdot \frac{T}{E_s}$ $k = 4\pi^2$ for ellipse $k = 40$ to 42 for rectangles

T = Twisting moment. ϕ = Angle of twist per unit of length. a = Area of cross-section. J = Polar moment of inertia. E_s = Shearing modulus of elasticity.

the values of α given in the following table will make the values from the equation fit closely those given by Saint Venant's rigorous analysis.

$\frac{h}{b} =$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4	5	10	∞
$\alpha =$	.208	.231	.246	.256	.267	.282	.292	.312	.333

Table 10 gives values for the maximum torsional shearing stress and the angle of twist in bars of several different forms of cross-section.

PROBLEMS

463. A steel bar having a rectangular cross-section $\frac{1}{2}$ in. wide and $1\frac{1}{2}$ in. long is subjected to a twisting moment of 1000 lb.-in. The shearing proportional limit of the material is 12,000 lb. per sq. in. (*a*) Calculate the maximum shearing unit-stress and state where in the bar it occurs. (*b*) Calculate the shearing unit-stress at the center of the short side. Use eq. (301) and compare with (300).

464. A bar of spring steel has a tensile proportional limit of 50,000 lb. per sq. in. and a shearing proportional limit equal to 0.6 of the tensile. The bar has a rectangular cross-section and is subjected to a twisting moment of 5000 lb.-in. If the working stress is two-thirds of the shearing proportional limit, and the width of the section is $\frac{3}{4}$ in., what should be the length of the section? *Ans.* 2 in., from eq. (300).

465. A rectangular bar having a cross-section such that $\frac{h}{b} = k$, and a cylindrical bar having a diameter d are subjected to the same twisting moment T. Show that when

$$d = 1.042b\sqrt[3]{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (302)$$

the maximum shearing unit-stresses in the two bars are equal.

466. If the cross-section of a rectangular bar is $1\frac{1}{2}$ in. thick, what must be its width in order that the maximum shearing stress produced in it shall be the same as that in a cylindrical shaft 2 in. in diameter, both shafts being subjected to the same twisting moment?

467. Two bars, one with a square cross-section and the other with a circular cross-section, have equal cross-sectional areas and are subjected to equal twisting moments. Show that the maximum torsional shearing unit-stress in the square bar is 1.27 times that in the circular shaft.

CHAPTER XX

COMPOSITE BEAMS. REINFORCED CONCRETE BEAMS

154. Composite (Two-material) Beam.—In the derivation of the flexure formula $M = \frac{sI}{c}$ it was assumed that the material of the beam was homogeneous. The formula, therefore, does not apply directly to a composite beam as, for example, a two-material beam such as a timber beam reinforced by steel plates, or a reinforced concrete beam. It is sometimes convenient, however, to transform the section of such a beam

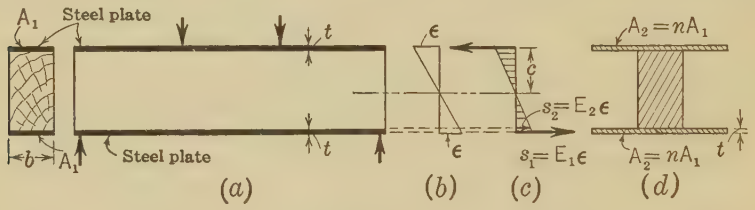


FIG. 328.—A beam reinforced on top and bottom.

to an equivalent one-material (homogeneous) section and to apply the flexure formula.

155. Timber Beam Reinforced by Steel Plates.—*Plate on Both Tension and Compression Faces.*—Let it be required to find the unit-stress in the timber and in the steel plates at any section of the beam shown in Fig. 328, in which the thickness of the plates is small compared to the depth of the beam. The plates are attached to the timber so that the timber and plates act together as a unit. In other words, it is assumed that plane sections remain plane after bending the same as in homogeneous beams, that is, the unit-strains of the fibers vary directly as the distances of the fibers from the neutral axis as indicated in Fig. 328*b*. Further, since the plates are thin, the unit-strains of the steel plates and of the outer fibers of the timber at any section are nearly equal—denoted by ϵ in (*b*) of Fig. 328. The unit-stress in the steel, however, is much greater than that in the outer fibers of the timber, owing to the difference in the moduli of elasticity of the two materials. The unit-stress in the steel is $E_1\epsilon$ and that in the outer fibers of the

timber $E_2\epsilon$, where E_1 and E_2 are the moduli of elasticity of the steel and timber, respectively.

The total stress in each steel plate then is $E_1\epsilon A_1$, where A_1 is the area of cross-section of one steel plate. If, now, the steel is replaced by timber so that the total stress in the added timber is equal to that in the steel, and acts at the same distance from the neutral axis, we shall have a one-material or homogeneous beam having the transformed section shown in Fig. 328(*d*). The total stress in each added timber area A_2 is $E_2\epsilon A_2$, hence

$$E_2\epsilon A_2 = E_1\epsilon A_1 \quad \text{or} \quad A_2 = \frac{E_1}{E_2}A_1 = nA_1 \quad . \quad . \quad (303)$$

Thus the area of the timber to be added in place of the steel is n times the area replaced, where n is the ratio of the modulus of elasticity of the material replaced to that of the material added. The flexure formula may now be applied to the homogeneous beam having this transformed section. The solution of this simple problem, however, may be made more conveniently perhaps from Fig. 328(*c*) than from the transformed section.

ILLUSTRATIVE PROBLEM

Problem 468.—Let the beam in Fig. 328(*a*) be a simple beam with a span of 12 ft. supporting a concentrated load of 4000 lb. at the center of the span. Let the width of the beam be 4 in. and its depth 6 in. and let the thickness of each steel plate be $\frac{1}{8}$ in. Calculate the maximum unit-stress in the steel and in the timber. The moduli of elasticity of the timber and steel are 10^6 and 30×10^6, respectively; therefore $n = 30$.

Solution.—s_1 = unit-stress in steel, s_2 = unit-stress in outer fiber of timber, $s_1 = ns_2 = 30s_2$.

Bending moment = Resisting moment

$$2000 \times 72 = \frac{s_2 I}{c} + s_1 A_1 \times 6\tfrac{1}{8}$$

$$= \frac{s_2 \frac{1}{12} 4 \times 6^3}{3} + 30s_2 \times 4 \times \tfrac{1}{8} \times 6\tfrac{1}{8}$$

$$= 24s_2 + 91.8s_2 = 115.8s_2$$

Therefore, $s_2 = 1240$ lb./in.^{2} and $s_1 = 30s_2 = 37{,}200$ lb./in.^{2} Or, instead of obtaining the resisting moment in the timber from $\frac{s_2 I}{c}$, it may be found as the moment $T_2 \times \frac{2}{3}d$ (or $C_2 \times \frac{2}{3}d$) of the couple T_2 and C_2, as indicated in Fig. 329, where T_2 is the total tensile stress in the timber below the neutral axis. Thus the moment in the timber is

$$(\tfrac{1}{2}s_2 \times \tfrac{1}{2}bd) \times \tfrac{2}{3}d = \tfrac{1}{6}s_2 bd^2 = 24s_2.$$

Since the usual working stresses for timber and structural steel are not greater than 1000 lb./in.² and 20,000 lb./in.², the results show that both materials are overstressed, particularly the steel.

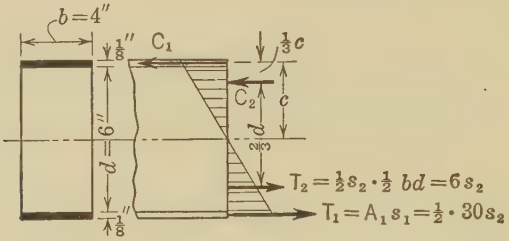


FIG. 329.

PROBLEMS

469. Solve Prob. 468 using the method of the transformed section.

470. A rectangular timber member having a width of 6 in. and a depth of 10 in. is used as a horizontal cantilever beam having a free length of 8 ft. Steel plates ¼ in. thick are rigidly attached to the top and bottom faces of the beam. The beam supports a uniformly distributed load. If the maximum unit-stress in the timber is 1000 lb./in.², what is the maximum unit-stress in the steel and what is the total load on the beam? The modulus of elasticity for the timber is 2×10^6 and for steel 30×10^6 lb./in.² *Ans.* 6890 lb.

471. If in Prob. 468 steel plates ¼ in. thick are attached to the vertical sides of the timber beam instead of the top and bottom faces, what will be the maximum unit-stress in the steel and in the timber? The steel plates cover the sides of the beam completely.

472. Assuming a working stress of 600 lb./in.² for timber, determine the proper thickness of the steel plates for the timber beam of Prob. 468. Use $n = 30$.

Ans. $t = 0.285$ in.

Steel Plate on Tension Side Only.—Let it be required to determine the maximum unit-stress in the steel plate and in the outer fibers of the

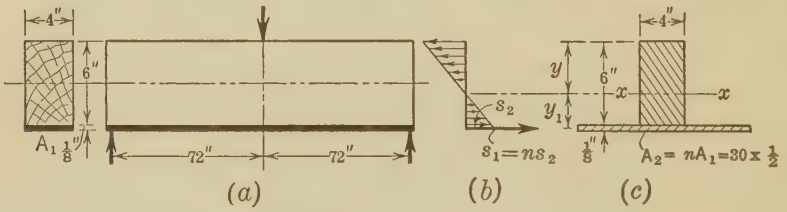


FIG. 330.—A beam reinforced on bottom side.

timber at any section of the beam shown in Fig. 330. Since the timber and steel are assumed to act together as a beam, the unit-strains of the fibers vary directly as the distances of the fibers from the neutral

axis of the beam. But the location of the neutral axis is not known; it may be found conveniently by locating the centroidal axis of the transformed section since the neutral axis and centroidal axis are coincident in a homogeneous beam (Art. 31).

The transformed section is obtained by replacing the area A_1 of the steel by an area A_2 equal to nA_1 as discussed above, where n is the ratio of the modulus of elasticity of the material replaced to that of the material added. The transformed section is shown in Fig. 330(c). The stress in the beam with this section may be found from the flexure formula.

ILLUSTRATIVE PROBLEM

Problem 473.—In Prob. 468 let it be assumed that the steel plate is removed from the top face of the beam leaving the beam reinforced only on the bottom or tension face. Let it be required to find the maximum moment the beam can resist without causing a stress in the timber greater than 1000 lb./in.^{2} What will be the corresponding stress in the steel?

Solution.—The transformed section is shown in Fig. 330(c). The neutral axis XX of the beam is the centroidal axis of this area which is at the distance y from the top of the beam. By taking moments of areas about the top of the section, y may be found as follows:

$$(24 + 15)y = 24 \times 3 + 15 \times 6.06. \quad \text{Hence} \quad y = 4.18 \text{ in.}$$

$$I_x = \tfrac{1}{12} 4 \times 6^3 + 24 \times (1.18)^2 + 15(6.06 - 4.18)^2$$

$$= 72 + 33.5 + 53.1 = 158.6 \text{ in.}^4$$

$$M = \frac{sI_x}{y} = \frac{1000 \times 158.6}{4.18} = 38{,}000 \text{ lb.-in.}$$

Stress in added timber area is $s_2 = \dfrac{y_1}{y} s_1 = \dfrac{1.88}{4.18} 1000 = 450$ lb./in.^{2}

Therefore the stress in the steel is $s_1 = 450 \times 30 = 13{,}500$ lb./in.^{2}

156. Reinforced Concrete Beams.—Concrete is weak in tension and relatively strong in compression, therefore when a beam is made of concrete, steel bars are usually embedded near the surface of the tension side of the beam in order to take the tensile stress, thereby making possible the effective use of the compressive strength of the concrete. Such a concrete beam is called a reinforced concrete beam, and the bars are called reinforcing bars. These bars may be either plain or deformed; a deformed bar is one having projections along the bar (produced by special rolling) which provide mechanical anchorage in addition to the ordinary adhesion or "bond" between the steel and concrete.

157. Stress in Steel and in Concrete.—Fig. 331 represents a rectangular reinforced concrete beam in which the steel is assumed to resist all the tensile stress; the concrete, when the beam is subjected to small loads, will offer tensile resistance but, when the beam is subjected to larger loads, the concrete will crack on the tensile side of the beam and practically all the tensile stress will then be carried by the steel bars. Hence it is customary to neglect the tensile resistance of the concrete in calculating the strength of the beam.

Let the depth of the center line of the bars below the top surface of the beam be denoted by d (Fig. 331), and let the depth of the neutral axis (unknown as yet) below the top surface be denoted by kd where k

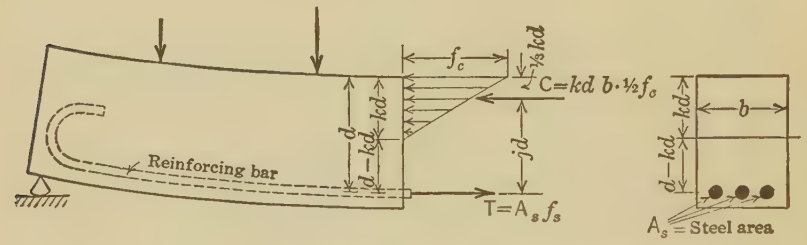


FIG. 331.—Stresses in reinforced concrete beam.

is less than unity; further let the moment-arm of the resisting couple be jd where j is less than unity. The meaning of k and j are very important in the study of this subject; values for them are determined below.

The cross-section of the beam is considered to be bd; the concrete below the center line of the reinforcement is neglected in the computations. Let the total cross-sectional area of the steel bars be denoted by A_s, and let the ratio $\frac{A_s}{bd}$ be denoted by p; this ratio is called the steel ratio or the percentage of steel. Thus, $p = \frac{A_s}{bd}$. Let the ratio of the modulus of elasticity, E_s, of the steel to that of the concrete, E_c, be denoted by n. Thus, $n = \frac{E_s}{E_c}$.

It is assumed that the compressive unit-stress in the concrete varies directly as the distance from the neutral axis; this assumption is approximately true for stresses that do not exceed the working stress, although for stresses near the ultimate strength of the concrete it is largely in error.

The stress[1] in the steel and in the concrete may now be expressed as follows. Since the beam (Fig. 331) is in equilibrium the total compressive stress C is equal to the total tensile stress T. But

T = area of steel $\times$ unit-stress in steel = $A_s f_s$,

and

C = compressive area $\times$ average unit-stress in concrete = $bkd \cdot \frac{1}{2} f_c$.

Hence,

$$A_s f_s = bkd \tfrac{1}{2} f_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (304)$$

in which f_s is the unit-stress in the steel and f_c is the unit-stress in the outer fibers of the concrete. Also from equilibrium the internal or resisting moment M_r must equal the external or bending moment M; that is, $M = M_r$. Now the resisting moment M_r is the moment of the couple formed by the total compressive stress C and the total tensile stress T; the moment of this couple is $C \cdot jd$ or $T \cdot jd$. Further, since the action line of C is at a distance of $\frac{1}{3}kd$ from the top surface of the beam (see Fig. 331) we have

$$jd = d - \tfrac{1}{3}kd, \quad \text{or} \quad j = 1 - \tfrac{1}{3}k \quad . \quad . \quad . \quad . \quad . \quad (305)$$

By substituting the value of T or of C in the above expression for the resisting moment and noting that $A_s = pbd$, the equation $M = M_r$ becomes

$$M = A_s f_s jd = p f_s j b d^2, \quad \text{or} \quad f_s = \frac{M}{pjbd^2} \quad . \quad . \quad . \quad (306)$$

if the stress in the steel governs the design; and

$$M = \tfrac{1}{2} f_c k j b d^2, \quad \text{or} \quad f_c = \frac{M}{\frac{1}{2} jkbd^2} \quad . \quad . \quad . \quad . \quad . \quad (307)$$

if the stress in the concrete governs.

Location of Neutral Axis.—But k (and therefore j also) is unknown, and hence the position of the neutral axis must be found before the above equations can be used for finding the stress in the steel and the concrete. The neutral axis may be found by determining the centroidal axis of the transformed section as was done in the preceding article.

The transformed section is shown as the cross-hatched area in Fig. 332 in which the tensile area of the concrete below the neutral axis is

[1] The symbol f will be used to denote unit-stress in this and the following articles in conformity with the usual practice of treating stresses in reinforced concrete; and the subscripts s and c will be used to denote the stress in the steel and concrete, respectively.

neglected and the steel area A_s is replaced by an area of concrete equal to nA_s, where n is the ratio of the modulus of elasticity of the replaced material (steel) to that of the added material (concrete).

In any numerical problem the value of kd may easily be computed by locating the centroidal axis of the transformed section. Or a literal expression for it may be obtained as follows. Since the moment of the cross-hatched areas (Fig. 332) about the neutral axis must be equal to zero we have

$$bkd \times \tfrac{1}{2}kd = nA_s \times (d - kd) = npbd(d - kd)$$

Or,

$$k^2 + 2pnk - 2pn = 0,$$

whence,

$$k = \sqrt{2pn + (pn)^2} - pn \quad . \quad . \quad . \quad (308)$$

Value of n.—The value of n is usually taken to be 15 for concrete which is assumed to have an ultimate compressive strength of 2000 lb. per sq. in. and a modulus of elasticity of 2,000,000 lb. per sq. in. The value of n, however, for concrete having an ultimate compressive strength between 2200 and 2900 is usually assumed to be 12, and for concrete having an ultimate compressive strength greater than 2900 lb. per sq. in. n is assumed to be 10.

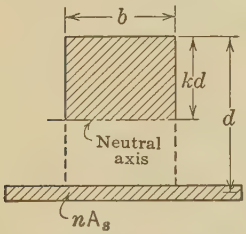


FIG. 332.—Transformed section.

Values of k and j.—For well-designed rectangular beams having tension reinforcement only, the values of k and j vary but little from $k = \frac{3}{8}$ and $j = \frac{7}{8}$. From equation (307), then, we have for approximate but reliable results

$$bd^2 = \frac{M}{\frac{1}{6}f_c} = \frac{6M}{f_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (309)$$

After thus determining bd^2 and assigning values to b and d the stress in the steel and the required steel area may be found from

$$T = \frac{M}{\frac{7}{8}d} \quad \text{and} \quad A_s = \frac{T}{f_s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (310)$$

Working Stress.—The Joint Committee on Concrete and Reinforced Concrete (composed of Members from several technical societies) recommend a value for the working stress in the extreme fiber of the concrete equal to 0.4 of the ultimate compressive strength of a test cylinder of the concrete ($0.4f'_c$) when 28 days old. Thus the working stress in

the extreme fiber of a beam made of concrete having an ultimate compressive strength, at 28 days, of 2000 lb. per sq. in. is 800 lb. per sq. in.

The working stress in the steel is usually taken to be 16,000 to 18,000 lb. per sq. in.

Steel Ratio.—The amount of reinforcement required to develop given working stresses in the concrete and the steel may be found by equating the expression for the bending moment based on the stress in the steel to that for the bending moment based on the stress in the concrete; that is, by equating values of M given by equations (306) and (307). The value of p thus found is

$$p = \frac{k}{2}\frac{f_c}{f_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (311)$$

Also, p may be expressed in terms of the working stresses and the ratio of the moduli of elasticity. Thus,

$$p = \frac{\frac{1}{2}}{\frac{f_s}{f_c}\left(\frac{f_s}{nf_c} + 1\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (312)$$

The value of p given by equations (311) and (312) is commonly termed the percentage for *balanced reinforcement*; if the reinforcement used is less than the percentage for balanced reinforcement, the beam may be said to be *under-reinforced*, and its strength is governed by the resisting moment of the steel. On the other hand, if the beam is *over-reinforced*, its strength is governed by the resisting moment of the concrete. It is well to note that the value of p for balanced reinforcement depends only on the values of the working stresses and n, that is, it is independent of the dimensions of the beam.

PROBLEMS

474. A concrete beam 10 in. wide and 20 in. deep is reinforced with four steel bars $\frac{3}{4}$ in. in diameter. The center of the steel area is $1\frac{3}{4}$ in. from the tension surface of the beam. The modulus of elasticity of the concrete is 2,000,000 lb. per sq. in. and that of steel is 30,000,000 lb. per sq. in. How far from the compressive surface of the beam is the neutral axis when the beam is subjected to its working load?

Ans. $kd = 7.5$ in.

475. Let the value of E_c in the above problem be 2,500,000 and find the value of kd.

476. A reinforced concrete beam is 10 in. wide and 14 in. deep to the center of the steel area; the area of the steel is 1.20 sq. in. $E_s = 30{,}000{,}000$ lb. per sq. in., and $E_c = 2{,}000{,}000$ lb. per sq. in. Find the maximum bending moment the beam can resist if the following values of working stresses are not to be exceeded; $f_c = 750$ lb. per sq. in. and $f_s = 16{,}000$ lb. per sq. in. *Ans.* $M = 233{,}000$ lb.-in.

477. The width of a reinforced concrete beam is 8 in. and the depth to the center of the steel area is 10 in. The beam is used with a span of 10 ft. and is subjected to a uniformly distributed load of 650 lb. per ft.; the area of the steel is 1.0 sq. in. Calculate the unit-stress in the steel and the maximum unit-stress in the concrete, using a value of 15 for n. *Ans.* $f_s = 11{,}500$ lb. per sq. in.; $f_c = 635$ lb. per sq. in.

478. What should be the cross-sectional area of the steel bars for a reinforced concrete beam having a width of 12 in. and a depth of 14 in. to the center of the steel area, if the values of f_c and f_s are 800 and 16,000 lb. per sq. in., respectively, and $E_c = 2{,}500{,}000$ lb. per sq. in.? *Ans.* $A_s = 1.58$ sq. in.

479. A reinforced concrete beam has a width of 10 in. The beam is used on a span of 16 ft., and is subjected to a total uniformly distributed load of 12,000 lb. including the weight of the beam. The ultimate compressive strength of the concrete is 1500 lb. per sq. in. What should be the depth of the beam? Use $f_s = 16{,}000$ lb./in.^{2} and $n = 15$. *Ans.* $d = 17.4$ in.

480. What should be the dimensions of a reinforced concrete beam to resist a bending moment of 60,000 lb.-ft. if $E_c = 2{,}000{,}000$ lb. per sq. in.; the working values of the stresses are $f_c = 600$ lb. per sq. in., $f_s = 14{,}000$ lb. per sq. in., and $n = 15$, and what should be the area A_s of the steel?

Ans. $bd^2 = 7060$ in.^{3}; $b = 14$ in., $d = 22.5$ in., $A_s = 2.64$ in.^{2}

158. Tee Beams.—A concrete floor system of the beam and slab type is usually considered to be composed of tee beams, the floor slab for some distance (usually limited by specifications) on each side of the beam forming the flange of the tee beam; the stem or web of the tee beam is that portion of the beam below the slab.

The formulas for the various quantities used in the design of tee beams are somewhat more complicated than those for rectangular beams. The procedure, however, is the same and will be illustrated in the solution of the following problem. The neutral axis of a tee beam may lie within the flange or within the stem. If the depth of the flange is $\frac{3}{8}d$ or more, the neutral axis will usually lie within the flange and such beams are treated the same as are rectangular beams, except for shearing stress, and will not here be considered.

ILLUSTRATIVE PROBLEM

Problem 481.—Let it be required to find the maximum bending moment that can be applied to the tee beam shown in Fig. 333 (a), assuming that the maximum compressive stress in the concrete is $f_c = 650$ lb. per sq. in.

Solution.—The transformed section is shown in (*b*) of Fig. 333. The distance kd from the top of this section to the centroid of the area is the distance to the neutral axis of the beam. This distance may be found by equating the sum of the moments of the areas (neglecting the tension area of the concrete) about the centroidal axis to zero. Thus (assuming $n = 15$).

$$10kd \cdot \tfrac{1}{2}kd + (60 - 10) \cdot 3.5 \cdot (kd - 1\tfrac{3}{4}) = 30\,(20 - kd)$$

$$kd = 4 \text{ in.}$$

If the area in the stem below the flange is neglected we have

$$(60 \times 3.5)\,(kd - 1.75) = 30\,(20 - kd)$$

$$kd = 4 \text{ in. (nearly)}$$

This shows that the area neglected has a negligible effect on the neutral axis. In fact the amount of compression in the stem below the flange in nearly all tee beams may be neglected without appreciable error.

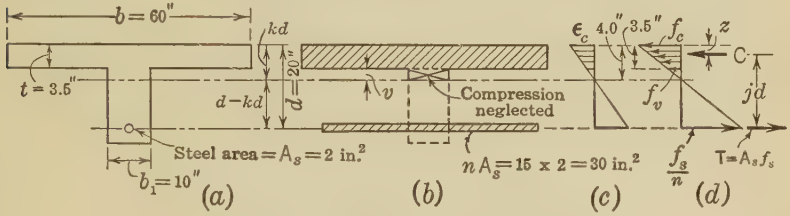


FIG. 333.—A reinforced concrete tee beam.

The resisting moment, which is equal to the bending moment, can be computed after the total compressive force C on the concrete and its action line are found: Neglecting the compression in the stem, we have

$$C = f_v \times 60 \times 3.5 + \tfrac{1}{2}(f_c - f_v)\,60 \times 3.5$$

$$= [\tfrac{1}{8}f_c + \tfrac{1}{2} \cdot \tfrac{7}{8}f_c]\,60 \times 3.5 = 26.2f_c + 91.8f_c = 118f_c$$

$$= 118 \times 650 = 76{,}700 \text{ lb.}$$

The action line of C may be found by taking moments about the top of the flange. Thus

$$(118f_c) \times z = 26.2f_c \times \tfrac{1}{2} \times 3.5 + 91.8f_c \times \tfrac{1}{3} \times 3.5 = 41.9f_c + 107f_c = 148.9f_c$$

$$z = 1.26 \text{ in.}$$

Therefore

$$jd = d - z = 20 - 1.26 = 18.7 \text{ in.} \quad \text{and} \quad j = \frac{18.7}{20} = 0.935.$$

Hence equating the bending moment M to the resisting moment we have

$$M = Cjd = 118f_c \times 18.7 = 2200f_c = 2200 \times 650 = 1{,}430{,}000 \text{ lb.-in.}$$

The total stress in the steel must be equal to C. Therefore,

$$A_s f_s = 118f_c \quad \text{or} \quad f_s = \frac{118 \times 650}{2} = 38{,}400 \text{ lb. per sq. in.}$$

Since the usual working stress for the steel is 16,000 to 20,000 lb. per sq. in. it is evident that the steel is greatly overstressed. Or, to state it differently, the steel area is too small to develop the full allowable stress of 650 lb. per sq. in. in the concrete. The proper amount of steel area is considered in the next problem.

It should be noted that T-beams are usually under-reinforced, which results in a low value of kd and a large value of jd. A value for jd of 0.92 is not unusual.

Problem 482.—Let it be required to find the steel area for the tee beam of Problem 481 for a given bending moment M of 100,000 lb.-ft. so that the steel will develop its full working value of $f_s = 16{,}000$ lb. per sq. in. and the concrete will not be stressed above its working value of $f_c = 650$ lb. per sq. in.

Solution.—The bending moment based on the stress in the steel is $M = T \cdot jd = A_s f_s \cdot jd$ and hence $A_s = \dfrac{M}{f_s jd}$. If then a value for jd can be found, the value of A_s can be calculated for the given M and f_s. An approximate value for jd may be found by assuming that the action line of the compressive force in the concrete is at the center of the flange, and hence jd equals the depth to the steel less half the flange thickness. This value of jd is smaller than the true value and hence errs on the side of safety since it makes C and T larger than their true values, but the error is not large. According to this assumption $jd = 20 - 1.75 = 18.25$.

Therefore,

$$A_s = \frac{M}{f_s jd} = \frac{100{,}000 \times 12}{16{,}000 \times 18.25} = 4.10 \text{ sq. in.}$$

The value of the stress f_c in the outer fibers of the flange is easily shown to be less than 650 lb. per sq. in., for f_c cannot be greater than twice the average unit stress in the concrete, and the average stress is

$$f_{av} = \frac{C}{60 \times 3.5} = \frac{T}{60 \times 3.5} = \frac{4.10 \times 16{,}000}{60 \times 3.5} = 310 \text{ lb. per sq. in.}$$

Formulas for Tee Beams.—By a procedure similar to that used in the above problems, employing literal instead of numerical quantities, the following formulas may be obtained (the compressive area of the stem is neglected).

$$kd = \frac{2ndA_s + bt^2}{2nA_s + 2bt}, \quad z = \frac{3kd - 2t}{2kd - t} \times \frac{t}{3}, \quad jd = d - z \quad . \quad (313)$$

$$f_c = \frac{Mkd}{bt\left(kd - \dfrac{t}{2}\right)jd} = \frac{f_s}{n}\frac{k}{l - k}, \quad k = \frac{1}{\dfrac{f_s}{nf_c} + 1} \quad . \quad . \quad . \quad (314)$$

The use of such formulas is simplified by the aid of curves giving values of k for various values of n and $\dfrac{f_s}{f_c}$, etc., which may be found in many books on reinforced concrete.

PROBLEM

483. Calculate the maximum safe bending moment that can be resisted by a tee beam, using the following data: $b = 48$ in., $t = 4$ in., $d = 22$ in. and $b_1 = 10$ in.; the limiting or working stresses are $f_c = 600$ lb. per sq. in., $f_s = 15{,}000$ lb. per sq. in.; $n = 15$ and $A_s = 2.65$ in.^{2}. *Ans.* 810,000 lb.-in.

159. Beams Reinforced in Both Tension and Compression.—If the concrete is reinforced on the compression side, the stress in the steel at any point on a given section will be n times the stress in the concrete at the same point. But since the cost of steel per unit-volume is greater than n times the cost of concrete, compression reinforcement is not as a rule economical. However, in cases where limitations are placed on the size of the beam, reinforcement on the compression side may be desirable.

The method of determining the amount of compression steel is illustrated in the following problem. Formulas giving the relations between the various quantities involved in double reinforced beams may be derived by the methods used in the solution of this problem and may be found in standard books on reinforced concrete.

ILLUSTRATIVE PROBLEM

Problem 484.—Let it be required to determine how much steel is needed for the beam in Fig. 334 assuming that it is to carry a maximum bending moment of 60,000 lb.-ft. and that the working stresses for steel and concrete are f_s = 16,000 lb. per sq. in. and f_c = 650 lb. per sq. in., and that n = 15.

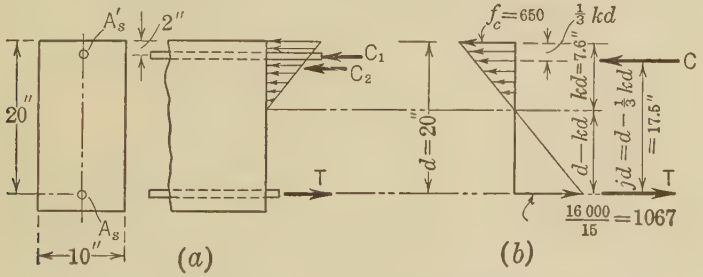


FIG. 334.—Beam reinforced in both tension and compression.

Solution.—The first step is to investigate the compressive resistance of the concrete and to see what compressive reinforcement is required. The resisting moment of the concrete, at a working fiber stress of 650 lb. per sq. in., is

$$M = \tfrac{650}{2} kjbd^2.$$

The value of kd is found from the transformed section or from the proportion (see Fig. 334*b*).

$$\frac{kd}{20 - kd} = \frac{650}{1067}; \quad kd = 7.60 \text{ in.} = 0.38d.$$

The value of jd so far as the centroid of the concrete stress is concerned is $20 - \frac{7.60}{3} =$ 17.5 in. as shown in Fig. 334*b*.

Hence, $M = \tfrac{650}{2} kjbd^2 = \tfrac{650}{2} \times .38 \times .875 \times 10 \times 400 = 432{,}200$ lb.-in. Since the total moment is $60{,}000 \times 12 = 720{,}000$ lb.-in., the compressive steel must

furnish 720,000 − 432,200 = 287,800 lb.-in. The compressive steel may be placed 2 in. from the top surface, so that its moment arm is 18 in. = 0.90d. Then its stress will be $650 \times 15 \times \frac{5.6}{7.6} = 7185$ lb. per sq. in. and the moment must be $7185A_c \times 18 = 287{,}800$ lb.-in. Thus the required compressive area is

$$A_c = \frac{287{,}800}{18 \times 7185} = 2.22 \text{ sq. in.}$$

The value of jd for the total resisting couple is between 17.5 and 18.0 in. It may be found by taking moments of the compressive forces about the line of action of T. Thus

$$40{,}685 \times jd = 17.5 \times 24{,}700 + 18.0 \times 15{,}985$$

$$jd = 17.7 \text{ in.}$$

The tensile steel area required is

$$A_s = \frac{M}{f_s jd} = \frac{720{,}000}{16{,}000 \times 17.7} = 2.54 \text{ sq. in.}$$

PROBLEM

485. Calculate the areas for the tensile and compressive steel for the beam shown in Fig. 335. Assuming the working stresses $f_c = 500$ lb. per sq. in. and $f_s = 15{,}000$ lb. per sq. in., a bending moment of 90,000 lb.-ft. and $n = 15$.

Ans. $A'_s = 3.6$ in.^{2}; $A_s = 3.0$ in.^{2}

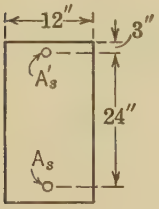


FIG. 335.

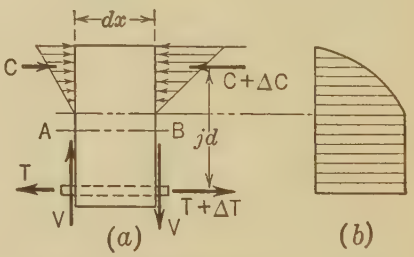


FIG. 336.—Shearing stress in beam.

160. Shearing Stress.—Shearing stress in reinforced concrete beams is of importance mainly in connection with diagonal tensile stress. Let it be required to determine the shearing unit-stress in a reinforced concrete beam. In Fig. 336 is represented a portion of a beam between two sections a short distance dx apart, the width of the beam being b. In general, the bending moment varies along the beam, the moment being M at one section and $M + \Delta M$ at a section the small distance dx away, causing stresses at the two sections shown in the figure. Since the sections are a very short distance apart, the total vertical shearing stresses V on the two sections are assumed to be equal.

The forces shown in the figure hold this small portion of the beam in equilibrium and hence the equilibrium equation $\Sigma M = 0$ applies. Thus

$$\Delta T \times jd = V \times dx \quad \text{or} \quad \Delta T = \frac{V dx}{jd} \quad . \quad . \quad (315)$$

The horizontal shearing unit-stress on any section between the neutral axis and the steel reinforcing bars (assuming the concrete to take no tensile stress is)

$$v = \frac{\Delta T}{bdx} = \frac{V}{bjd} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (316)$$

As noted in Art. 157 the value of j does not vary greatly for a wide range of conditions and an average value of $\frac{7}{8}$ is usually taken for all shearing computations. Hence, approximately,

$$v = \frac{8}{7}\frac{V}{bd} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (317)$$

Above the neutral axis the horizontal (and vertical) shearing unit-stress varies as in a homogeneous beam (see Art. 38) and hence the shearing unit stress at any section is as represented in Fig. 336(*b*).

PROBLEMS

486. A concrete beam having tension reinforcement, only, has a width b of 18 in. and a depth to steel d of 23.5 in. (The percentage of steel is $p = 0.0107$.) The beam is used as a simple beam on a span of 40 ft. to support a uniformly distributed load of 24,000 lb. including the weight of the beam. Calculate the maximum shearing unit-stress in the beam and state where in the beam it occurs. Use $n = 15$.

Ans. $v = 33$ lb. per sq. in.

487. The allowable shearing unit-stress in a concrete beam made of a 1 : 2 : 4 mix is frequently specified to be 40 lb. per sq. in. Calculate the maximum concentrated load that could be applied to the beam described in the preceding problem (neglecting the weight of the beam) without causing a shearing stress greater than this specified value.

161. Bond Stress and Anchorage.—In all the preceding discussion it has been assumed that the concrete and steel act together as a unit; that is, no slipping between the steel and concrete occurs. The result of such slipping may be undesirable cracking of the concrete and even failure of the beam.

The stress on the surface of contact between the steel reinforcing bars and the concrete is called the bond stress. The total bond stress between any two sections along a bar on the tensile side of a beam is equal to the difference $T_1 - T_2$ of the total tensile stresses in the bar

at the two sections. If x denotes the distance between the sections we have:

$$\text{Total bond stress per unit length} = \frac{T_1 - T_2}{x}.$$

But from eq. (316),

$$\frac{T_1 - T_2}{x} = vb.$$

Now if u denotes the bond unit-stress and Σo the sum of the bar perimeters we have

$$u = \frac{vb}{\Sigma o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (318)$$

Embedment and Anchorage.—To prevent slipping any bar must extend (be embedded) beyond any section a distance sufficient to develop a total bond stress equal to the total tensile stress that exists in the bar at the given section. The length of embedment becomes particularly important in supplying sufficient anchorage at the ends of members as, for example, at the end of a beam that is attached to a column, and at sections near the ends of beams where some of the longitudinal bars are stopped because they are not needed to take the reduced bending moment near the ends of the beam. For example, if the bars in a beam (Fig. 337) have a total stress P at section AA and some of the bars are not needed to the left of this section, those bars that are not needed should be embedded a length L such that

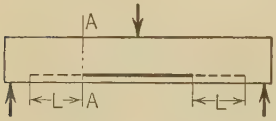


Fig. 337.—Embedment to secure bond stress.

$$u \cdot \Sigma o \cdot L = P$$

or for a single bar of diameter d

$$u \cdot \pi d \cdot L = f_s \tfrac{1}{4}\pi d^2 \quad \text{or} \quad L = \frac{f_s}{4u} d \quad . \quad . \quad . \quad . \quad (319)$$

Another way to secure anchorage is to bend the end of the bar to form a hook (see Fig. 331). The bend should be of ample radius and length to cause the elastic limit of the steel to be developed before crushing of the concrete occurs under the hook; a total length of hook of 16 bar diameters and a semi-circular hook having a radius of 4 bar diameters are frequently specified. Mechanical devices such as nuts and washers are sometimes used to provide anchorage mainly as insurance against

failure if the bond between steel and concrete proves insufficient to prevent slipping.

162. Diagonal Tension.—In Art. 87 the nature of the diagonal tensile stress in a beam is briefly discussed. It is important to note that near the ends of a simple beam, although the longitudinal bending stress $s = \frac{Mc}{I}$ is small (because M is small), the shearing unit-stress is high, and hence the diagonal tensile stress in the lower portion of the beam is not small. Furthermore, in a reinforced concrete beam much larger shearing stresses exist immediately above the steel than would occur at the same place in a honogeneous beam.

In a beam reinforced with longitudinal bars, only, the cracks that occur on the tension side in the central portion of the beam are nearly vertical and involve no danger since their opening is limited by the small deformation of the steel. Toward the ends of the beam, however, where the cracks are inclined (see Figs. 230, 338) the longitudinal reinforcing bars are not at right angles to the cracks and a serious condition arises which may be explained briefly as follows: The opening of a diagonal crack such as AB in Fig. 338 is accompanied by a large pressure on the bars at C and this in turn causes a horizontal crack followed by tearing of the bars from the concrete along the surface DE, resulting in failure of the beam. It is clear then that reinforcement in addition to longitudinal bars is needed for diagonal tension.

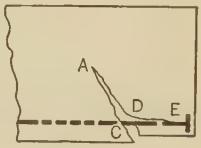


FIG. 338. — Failure due to lack of diagonal tension reinforcement.

163. Reinforcement for Diagonal Tension.—Bars at right angles to the cracks are the ideal reinforcement for diagonal tension. Vertical members, called stirrups, however, are usually more convenient to place and are reasonably effective.

In estimating the stresses in the reinforcement for diagonal tension it is usually assumed that at any point below the neutral axis the horizontal (and vertical) shearing unit-stress is approximately equal to the diagonal tensile stress at the point and hence is given by the equation $v = \frac{V}{bjd} = \frac{8}{7}\frac{V}{bd}$ (see eq. 317). This assumption holds primarily for a point on the neutral axis (see Fig. 229) but in the portion of the beam where diagonal tension is of importance the shearing stress is the dominant factor since the longitudinal bending stress there is small.

Further, it is usually assumed that the concrete can be relied on to resist a working shearing unit-stress (and hence diagonal tensile stress) equal to $0.02f'_c$ to $0.03f'_c$ where f'_c is the compressive ultimate strength

of the concrete at the age of 28 days, the smaller value applying to unanchored bars and the larger to well-anchored bars. The diagonal tensile unit-stress v' to be taken by the steel is thus

$$v' = v - 0.02f'c \quad \text{or} \quad v' = v - 0.03f'c \qquad . \; . \quad (320)$$

The relation $v' = \frac{2}{3}v$ is also sometimes used.

Two forms of diagonal reinforcement are here considered; namely, vertical stirrups (Fig. 339*a*) and rods inclined 45° (Fig. 339*b*) which may be either inclined stirrups or bent-up longitudinal bars.

For both forms of reinforcement let it be assumed that the diagonal tensile crack will occur on a 45-degree plane as indicated in Fig. 339,

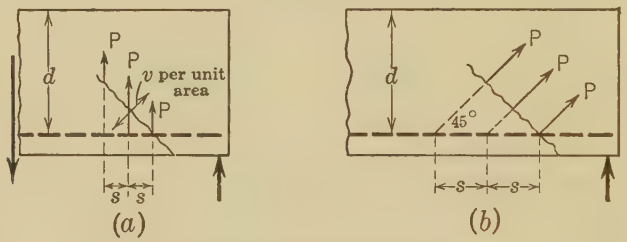


Fig. 339.—Diagonal tension reinforcement.

and let s denote the spacing of the diagonal reinforcing rods along the horizontal.

For vertical stirrups (Fig. 339*a*) the spacing along the crack is s sec 45° and the area reinforced by one rod or stirrup is bs sec 45° where b is the width of the beam. Since the diagonal tensile unit-stress in the concrete is considered to be v, the total tensile stress on this area is vbs sec 45° and the part of this to be taken by the reinforcement is $v'bs$ sec 45°. The vertical component of this is $v'bs$. If then P denotes the allowable total stress in one vertical stirrup, the spacing of the stirrup is given by

$$P = v'sb \quad \text{or} \quad s = \frac{P}{v'b} \qquad . \; . \; . \; . \; . \; . \; . \quad (321)$$

Likewise for stirrups inclined 45° or bent-up bars (Fig. 339*b*) the spacing along the crack is s cos 45° and the total tensile stress in this length is vbs cos 45°. The amount of this to be taken by the reinforcement is $v'bs$ cos 45°. Thus if P is the allowable total stress in one bar we have

$$P = 0.707v'bs \quad \text{or} \quad s = \frac{1.41P}{v'b} \qquad . \; . \; . \; . \; . \quad (322)$$

If the inclined rods make an angle θ with the horizontal, the result is easily found to be

$$s = \frac{P(\sin\theta + \cos\theta)}{v'b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (323)$$

PROBLEM

488. The stirrups in the beam shown in Fig. 340 are spaced 4 in. apart and are assumed to be stressed to 10,000 lb. per sq. in. when resisting all the diagonal tensile

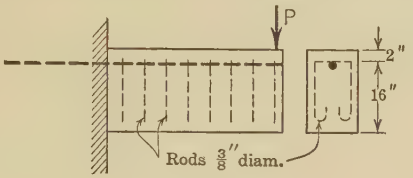


FIG. 340.

stress. Calculate the value of the load P considering the weight of the beam to be negligible. Assume j to be $\frac{7}{8}$ (see Art. 157). *Ans.* 7730 lb.

REFERENCES FOR FURTHER STUDY

1. Sutherland and Clifford. "Reinforced Concrete Design." John Wiley & Sons.

2. Turneaure and Maurer. "Principles of Reinforced Concrete Construction," 4th Edition. John Wiley & Sons.

APPENDIX I

PROPERTIES OF AN AREA

§ 1. First Moments and Centroids of Areas

164. Definitions.—The moment of an area with respect to an axis is the algebraic sum of the moments of the elementary parts of the area, the moment of each part being the product of the elementary area and the perpendicular distance from the elementary area to the axis. This moment is called the first moment to distinguish it from the moment of inertia (or second moment, see next section) of the area. The first moment of an area is sometimes called the statical moment of the area.

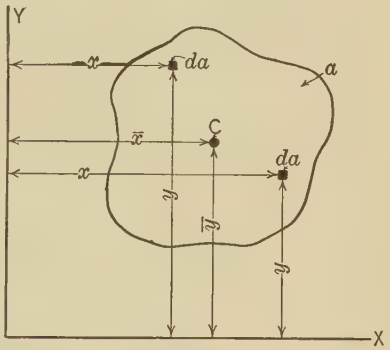


FIG. 341.

The centroid of an area is a point whose distance (called a centroidal distance) from any axis times the total area is equal to the moment of the area with respect to that axis. Hence, the coordinates x and y, of the centroid C of an area a (Fig. 341) may be found from the equations

$$a\bar{x} = \int xda \quad \text{and} \quad a\bar{y} = \int yda. \quad . \quad . \quad . \quad . \quad (324)$$

Or, expressed in another way, the centroid of an area is that point at which the whole area may be conceived to be concentrated and have the same moment with respect to any axis as has the actual (distributed) area.

If an area is symmetrical with respect to an axis, the centroid of the area lies on the given axis. This statement is evident from the fact that the moments of the areas on the opposite sides of the axis are numerically equal but of opposite sign. If an area is symmetrical with respect to each of two axes, the centroid of the area is the point of intersection of the two axes.

165. Centroids Found by Integration.—In determining the centroid of an area by the method of integration, from the equations in the preceding article, it is possible to select the element of area in various ways and to express the element in terms of either Cartesian or polar coordinates. The resulting integral may be a single or a double integral, depending on the way the element is selected. The integral, of course, is a definite integral, the limits of integration depending on the boundary curve of the area. In any case the element of area must be taken so that:

1. All points of the element are the same distance from the line about which moments are taken; otherwise, the distance from the line to the element will be indefinite.

2. The centroid of the element is known, in which case the moment of the element about the moment axis is the product of the element and the distance of its centroid from the axis or plane.

The centroids of some of the common areas will be found in the following illustrative problems:

ILLUSTRATIVE PROBLEMS

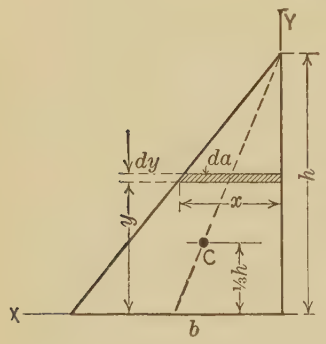


Fig. 342.

Find, by the method of integration, the centroids of the following areas with respect to the axes indicated.

Problem 489.—Area of a Triangle.—In accordance with the first of the above rules, the elements of area will be taken as strips parallel to the base of the triangle (Fig. 342). Since each element is bisected by the median drawn from the vertex opposite the base, the centroid of each element, and hence of the area, lies on this median. If x denotes the width of the strip, the area of the strip is $da = xdy$. Thus,

$$a\bar{y} = \int xydy.$$

From similar triangles, the relation between x and y is

$$\frac{x}{h-y} = \frac{b}{h} \quad \text{or} \quad x = \frac{b}{h}(h-y).$$

Hence

$$a\bar{y} = \frac{b}{h}\int_0^h (h-y)ydy = \tfrac{1}{6}bh^2.$$

Therefore,

$$\bar{y} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh} = \tfrac{1}{3}h.$$

The centroid of a triangular area, then, is on the median line at a distance of one-third of the altitude from the base.

Problem 490. Sector of a Circle. *First Method.*—The element of area will be selected in accordance with the first of the above rules as indicated in Fig. 343.

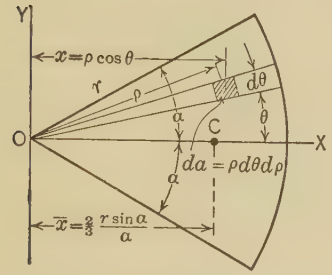


FIG. 343.

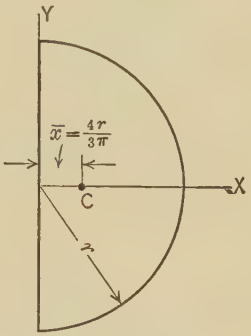


FIG. 344.

Since the area is symmetrical with respect to the x-axis, the centroid lies on this axis, and hence $\bar{y} = 0$. The value of $\bar{x}$ may then be found from the equation,

$$a\bar{x} = \int x\,da = \int_0^r \int_{-\alpha}^{+\alpha} \rho\cos\theta\cdot\rho\,d\rho\,d\theta = \tfrac{2}{3}r^3\sin\alpha.$$

Therefore,

$$\bar{x} = \frac{\frac{2}{3}r^3\sin\alpha}{\alpha} = \frac{\frac{2}{3}r^3\sin\alpha}{r^2\alpha} = \tfrac{2}{3}\frac{r\sin\alpha}{\alpha}.$$

If $\alpha = 90° = \dfrac{\pi}{2}$ radians, that is, if the sector is a semicircular area, $\bar{x} = \dfrac{4r}{3\pi}$ as indicated in Fig. 344.

Second Method.—In accordance with the second of the above rules, the element of area will be selected as a triangle, as indicated in Fig. 345. The area of the triangle is $\frac{1}{2}r^2d\theta$ and the distance of its centroid from the y-axis is $\frac{2}{3}r\cos\theta$. Hence, the moment of the triangle with respect to the y-axis is $\frac{1}{3}r^3\cos\theta d\theta$, and $\bar{x}$ is obtained from the equation

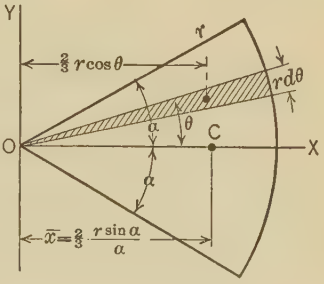


FIG. 345.

$$a\bar{x} = \int_{-\alpha}^{+\alpha} \tfrac{1}{3}r^3\cos\theta d\theta = \tfrac{2}{3}r^3\sin\alpha.$$

Therefore,

$$\bar{x} = \frac{\frac{2}{3}r^3\sin\alpha}{r^2\alpha} = \frac{2}{3}\frac{r\sin\alpha}{\alpha}.$$

Problem 491. Area of Quadrant of an Ellipse.—The semi-axes of the ellipse will be denoted by b and h (Fig. 346), and hence the equation of the ellipse is

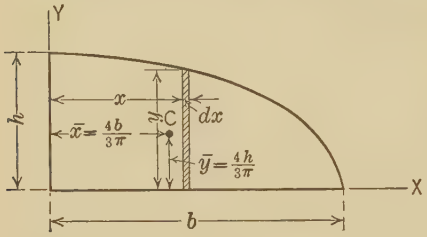


Fig. 346.

$$\frac{x^2}{b^2}+\frac{y^2}{h^2}=1.$$

A strip parallel to the y-axis will be selected as the element of area. From the equation of the ellipse, y may be expressed in terms of x by the equation

$$y=\frac{h}{b}\sqrt{b^2-x^2}.$$

Hence

$$a=\int_0^b y\,dx=\frac{h}{b}\int_0^b\sqrt{b^2-x^2}\,dx=\tfrac{1}{4}\pi bh.$$

To find $\bar{x}$:

$$\tfrac{1}{4}\pi bh\cdot\bar{x}=\int x\,da=\int_0^b x\cdot y\,dx=\frac{h}{b}\int_0^b x\sqrt{b^2-x^2}\,dx=\tfrac{1}{3}b^2h.$$

Therefore,

$$\bar{x}=\frac{\frac{1}{3}b^2h}{\frac{1}{4}\pi bh}=\frac{4b}{3\pi}.$$

To find $\bar{y}$, the same strip will be used for the element of area, its centroid being at the distance $\frac{1}{2}y$ from the x-axis. Thus,

$$\tfrac{1}{4}\pi bh\cdot\bar{y}=\int_0^a\tfrac{1}{2}y\,y\,dx=\frac{h^2}{2b^2}\int_0^b(b^2-x^2)dx=\tfrac{1}{3}bh^2.$$

Therefore,

$$\bar{y}=\frac{\frac{1}{3}bh^2}{\frac{1}{4}\pi bh}=\frac{4h}{3\pi}.$$

Problem 492. Parabolic Segment.—Let the segment be bounded by the x-axis the line $x=b$, and the parabola $y^2=\dfrac{h^2x}{b}$ as shown in Fig. 347. A strip parallel to the y-axis will be selected as the element of area, the area of the strip being expressed by ydx. The area of the segment, then, is

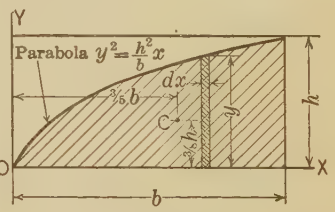


Fig. 347.

$$a=\int da=\int_0^b y\,dx=\frac{h}{\sqrt{b}}\int_0^b\sqrt{x}\,dx$$

$$=\frac{h}{\sqrt{b}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^b=\tfrac{2}{3}bh.$$

To find $\bar{x}$:

$$a\bar{x}=\int x\,da=\int_0^b xy\,dx=\frac{h}{\sqrt{b}}\int_0^b x^{\frac{3}{2}}dx=\tfrac{2}{5}hb^2.$$

Therefore,

$$\bar{x} = \frac{\frac{2}{5}hb^2}{a} = \frac{\frac{2}{5}hb^2}{\frac{2}{3}bh} = \frac{3}{5}b.$$

To find $\bar{y}$, the same elementary strip will be selected, but, since each point of the element is not the same distance from the x-axis, its moment must be expressed as the product of the area of the strip and its centroidal distance, $\frac{y}{2}$, from the x-axis. Thus,

$$a\bar{y} = \int_0^b \frac{1}{2}y \cdot y\,dx = \frac{1}{2}\frac{h^2}{b}\int_0^b x\,dx = \frac{1}{4}h^2b.$$

Therefore,

$$\bar{y} = \frac{\frac{1}{4}h^2b}{a} = \frac{\frac{1}{4}h^2b}{\frac{2}{3}bh} = \frac{3}{8}h.$$

Problem 493. Shaded Area in Fig. 348.—Let it be required to locate the centroid of the area between the y-axis, the line $y = h$, and the parabola $y^2 = \frac{h^2x}{b}$ represented by the shaded area in Fig. 348.

The element of area shown in the figure is parallel to the x-axis, and is equal to xdy. The shaded area a is the area (bh) of the rectangle minus the area ($\frac{2}{3}bh$) of the parabolic segment. Hence $a = \frac{1}{3}bh$.

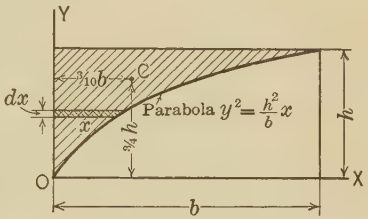


Fig. 348.

To find $\bar{x}$: In accordance with the second of the rules in Art. 165, we have

$$a\bar{x} = \int_0^h \left(xdy \cdot \frac{x}{2}\right) = \frac{1}{2}\int_0^h x^2dy$$

$$= \frac{1}{2}\int_0^h \frac{y^4b^2}{h^4}\,dy = \frac{1}{2}\frac{b^2}{h^4}\frac{h^5}{5} = \frac{1}{10}b^2h.$$

Therefore,

$$\bar{x} = \tfrac{3}{10}b.$$

To find $\bar{y}$:

$$a\bar{y} = \int yda = \int yxdy = \frac{b}{h^2}\int_0^h y^3dy = \frac{b}{h^2}\frac{h^4}{4} = \frac{bh^2}{4}.$$

Therefore,

$$\bar{y} = \tfrac{3}{4}h.$$

166. Centroids of Composite Areas.—As noted in Art. 164, if the centroid of an area is known, the moment with respect to an axis is most easily found by multiplying the area by the distance of the centroid from the axis. Thus, if a given area can be divided into parts, the centroids of which are known, the moment of the whole area may be found, without integrating, by obtaining the algebraic sum of the mo-

ments of the parts into which the area is divided, the moment of each part being the product of that part and the distance of its centroid from the line. Thus, for example, if a'_1, a'_2, a'_3, etc., denote the parts into which an area a is divided, and x'_0, x''_0, x'''_0, etc., denote the x-ordinates of the centroids of the respective parts, then

$$(a'_1 + a'_2 + a'_3 + \ldots)\bar{x} = a'_1x'_0 + a'_2x''_0 + a'_3x'''_0, \quad (325)$$

or,

$$a\bar{x} = \Sigma(a'x_0).$$

Similarly,

$$a\bar{y} = \Sigma(a'y_0).$$

ILLUSTRATIVE PROBLEM

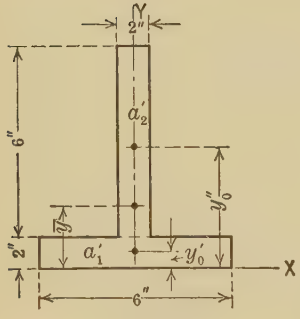


FIG. 349.

Problem 494.—Locate the centroid of the T-section shown in Fig. 349.

Solution.—If axes be selected as indicated, it is evident from symmetry that $\bar{x} = 0$. By dividing the given area into areas a'_1 and a'_2, and by taking moments about the bottom edge of the area, $\bar{y}$ may be found as follows:

$$a\bar{y} = \Sigma(a'y_0),$$

$$\bar{y} = \frac{12 \times 1 + 12 \times 5}{6 \times 2 + 6 \times 2} = 3 \text{ in.}$$

§ 2. Second Moment or Moment of Inertia of an Area

167. Moment of Inertia of an Area Defined.—In the analysis of many engineering problems as, for example, in determining the stresses in a beam or a shaft, expressions of the form $\int x^2 da$ are frequently met, in which da represents an element of an area a and x is the distance of the element from some axis in, or perpendicular to, the plane of the area, the limits of integration being such that each element of the area is included in the integration. An expression of this form is called the *second moment* of the area or the *moment of inertia* of the area with respect to the given axis.

The moment of inertia of an area with respect to an axis in, or perpendicular to, the plane of the area, may, then, be defined as the sum of the products obtained by multiplying each element of the area by the square of its distance from the given axis.

The term moment of inertia is somewhat misleading, since inertia is a property of physical bodies only, and hence an area does not possess inertia. For this reason the term, second moment of an area, is to be preferred, particularly when contrasting the expressions of the form here discussed with expressions which were defined as first moments of areas in the preceding section. It may be noted that each term x^2da in the summation can be written in the form $x(xda)$ and hence represents the moment of the moment of an element of area, that is, the second moment of the element. The term moment of inertia, however, is very widely used, on account of the fact that the expression is of the same form as an expression which is defined as the moment of inertia of a body and which does have a physical significance.

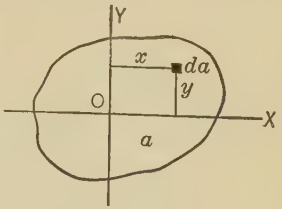


Fig. 350.

The moment of inertia of an area with respect to an axis will be denoted by I for an axis in the plane of the area and by J for an axis perpendicular to the plane of the area. The particular axis (or direction of the axis) about which the moment of inertia is taken will be denoted by subscripts. Thus, the moments of inertia of the area (Fig. 350) with respect to the x and y axes are expressed as follows:

$$I_x = \int y^2 da \quad \text{and} \quad I_y = \int x^2 da. \quad . \quad . \quad . \quad . \quad (326)$$

Units and Sign.—Since the moment of inertia of an area is the sum of a number of terms each of which is the product of an area and the square of a distance, the moment of inertia of an area is expressed as a length to the fourth power. If, then, the inch (or foot) be taken as the unit of length, the moment of inertia will be expressed as inches (or feet) to the fourth power (written in.^{4} or ft.^{4}). Further, the sign of each of the products x^2da is always positive, since x^2 is always positive, whether x is positive or negative, and da is essentially positive. Therefore the moment of inertia, or second moment, of an area is always positive. In this respect it differs from the first moment of an area, which may be positive, negative, or zero, depending on the position of the moment axis.

168. Polar Moment of Inertia.—The moment of inertia of an area with respect to a line perpendicular to the plane of the area is called the *polar moment of inertia* of the area and, as noted in Art. 167, will be denoted by J. Thus the polar moment of inertia, with respect to the z-axis, of an area in the xy-plane (Fig. 351) may be expressed as follows:

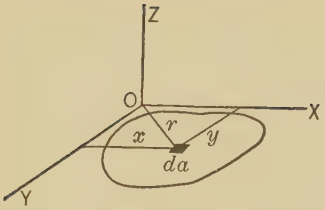


FIG. 351.

$$J_z = \int r^2 da$$

$$= \int (x^2 + y^2) da$$

$$= \int x^2 da + \int y^2 da.$$

Therefore,

$$J_z = I_y + I_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (327)$$

Hence the following proposition may be stated:

> The polar moment of inertia of an area with respect to any axis is equal to the sum of the moments of inertia of the area with respect to any two rectangular axes in the plane of the area which intersect on the given polar axis.

169. Radius of Gyration.—Since the moment of inertia of an area is four dimensions of length, it may be expressed as the product of the total area, a, and the square of a distance, k. Thus,

$$I_x = \int y^2 da = ak_x^2; \quad J_z = \int r^2 da = ak_z^2.$$

The distance k is called the *radius of gyration* of the area with respect to the given axis, the subscript denoting the axis with respect to which the moment of inertia is taken. The radius of gyration of an area with respect to a line, then, may be defined as a distance such that, if the area were conceived to be concentrated at this distance from the given line, the moment of inertia would be the same as the moment of inertia of the actual or distributed area with respect to the same line.

From the equation $I_y = \int x^2 da = ak_y^2$, it will be noted that k_y^2, the square of the radius of gyration with respect to the y-axis, is the mean of the squares of the distances, from the y-axis, of the equal elements of area into which the given area may be divided, and that it is not the square of the mean of these distances. The mean distance ($\bar{x}$) of the elements of area from the y-axis is the centroidal distance as discussed in the preceding chapter. Hence $a\bar{x}^2$ does *not* represent the moment of inertia of an area with respect to the y-axis.

170. Parallel Axis Theorem.—If the moment of inertia of an area with respect to a centroidal axis in the plane of the area is known, the moment of inertia with respect to any parallel axis in the plane may be determined, without integrating, by means of a proposition which may be established as follows: In Fig. 352, let YY be any axis through the centroid, C, of an area, and let $Y'Y'$ be any axis parallel to YY and at a distance d therefrom. Further, let the moment of inertia of the area with respect to the axis YY be denoted by I and the moment of inertia with respect to $Y'Y'$ by I'. By definition then,

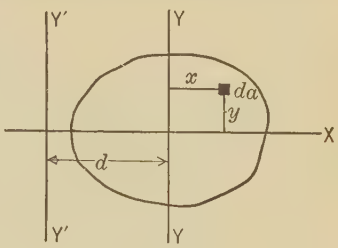


Fig. 352.

$$I' = \int (x + d)^2 da$$

$$= \int x^2 da + 2d \int x da + d^2 \int da.$$

Therefore,

$$I' = \bar{I} + ad^2, \text{ since } \int x da = a\bar{x} = 0. \quad . \quad . \quad . \quad (328)$$

Hence the following proposition may be stated:

> The moment of inertia of an area with respect to any axis in the plane of the area is equal to the moment of inertia of the area with respect to a parallel centroidal axis plus the product of the area and the square of the distance between the two axes. This proposition is called the parallel axis theorem.

A corresponding relation exists between the radii of gyration of the area with respect to two parallel axes, one of which passes through the centroid of the area. For, by replacing I' by ak^2 and $\bar{I}$ by $a\bar{k}^2$, the above equation becomes

$$ak^2 = a\bar{k}^2 + ad^2.$$

Whence,

$$k^2 = \bar{k}^2 + d^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (329)$$

where k denotes the radius of gyration of the area with respect to any axis in the plane of the area and $\bar{k}$ denotes the radius of gyration of the area with respect to a parallel centroidal axis.

Similarly, for polar moments of inertia and radii of gyration it can be shown that,

$$J = \bar{J} + ad^2,$$

and

$$k^2 = \bar{k}^2 + d^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (330)$$

where $\bar{J}$ and $\bar{k}$ denote the polar moment of inertia and radius of gyration, respectively, of the area with respect to the centroidal axis, and J and k denote the polar moment of inertia and radius of gyration, respectively, of the area with respect to an axis parallel to the centroidal axis and at a distance d therefrom.

171. Moments of Inertia Found by Integration.—In determining the moment of inertia of a plane area with respect to a line, it is possible to select the element of area in various ways and to express the area of the element in terms of either Cartesian or polar coordinates. Further, the integral may be either a single or double integral, depending on the way in which the element of area is selected; the limits of integration are determined, of course, from the boundary curve of the area. In any case, however, the elementary area must be taken so that:

(1) All points in the element are equally distant from the axis with respect to which the moment of inertia is to be found, otherwise the distance x in the expression x^2da would be indefinite. Or so that,

(2) The moment of inertia of the element, with respect to the axis about which the moment of inertia of the whole area is to be found, is known, the moment of inertia of the area then being found by summing up the moments of inertia of the elements. Or so that,

(3) The centroid of the element is known and also the moment of inertia of the element with respect to an axis which passes through the centroid of the element and is parallel to the given axis; the moment of inertia of the element may then be expressed by means of the parallel axis theorem.

The moments of inertia of some of the simple areas will now be found in the following illustrative problems:

ILLUSTRATIVE PROBLEMS

Problem 495.—Determine the moment of inertia of a rectangle, in terms of its base b and altitude h, with respect to (a) a centroidal axis parallel to the base; (b) an axis coinciding with the base.

Solution. (a) **Centroidal Axis.**—The element of area will be selected in accordance with rule (1) above, as indicated in Fig. 353. The moment of inertia of the rectangular area with respect to the centroidal axis, then, is,

$$\bar{I}_x = \int y^2 da = \int_{-h/2}^{+h/2} y^2 b dy = \tfrac{1}{12} b h^3.$$

(*b*) **Axis Coinciding with the Base.**—*First Method.*—The element of area will be selected as indicated in Fig. 354. The moment of inertia of the rectangle with respect to the base, then, is

$$I_b = \int y^2 da = \int_0^h y^2 b dy = \tfrac{1}{3} b h^3.$$

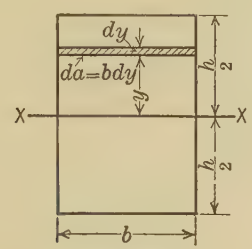


Fig. 353.

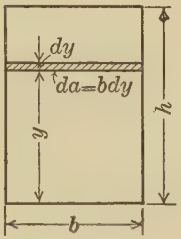


Fig. 354.

Second Method.—Since the moment of inertia of the rectangle with respect to a centroidal axis is $\frac{1}{12} b h^3$, the moment of inertia with respect to the base may be found from the parallel axis theorem (Art. 170). Thus,

$$I_b = \bar{I}_x + a\left(\frac{h}{2}\right)^2$$

$$= \tfrac{1}{12} b h^3 + bh \times \frac{h^2}{4} = \tfrac{1}{3} b h^3.$$

Problem 496.—Determine the moment of inertia of a triangle, in terms of its base b and altitude h, with respect to (*a*) an axis coinciding with its base; (*b*) a centroidal axis parallel to the base.

Solution. (*a*) **Axis Coinciding with the Base.**—The elementary area will be selected as shown in Fig. 355. The moment of inertia of the area of the triangle with respect to the base, then, is

$$I_b = \int y^2 da = \int y^2 x dy.$$

But, from similar triangles,

$$\frac{x}{b} = \frac{h - y}{h}, \quad \text{or} \quad x = \frac{b}{h}(h - y).$$

Therefore,

$$I_b = \frac{b}{h}\int_0^h y^2(h - y)dy = \tfrac{1}{12} b h^3.$$

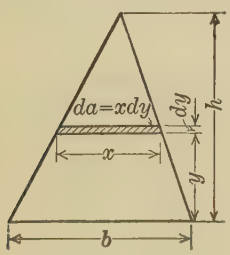


Fig. 355.

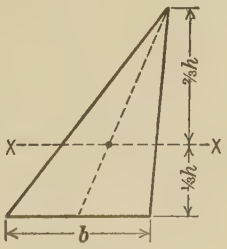


Fig. 356.

(*b*) **Centroidal Axis Parallel to the Base.**—The centroidal axis parallel to the base (axis XX) is shown in Fig. 356 (see Prob. 489). Using the parallel axis theorem, the moment of inertia of the triangular area with respect to the centroidal axis is

$$\bar{I}_x = I_b - a(\tfrac{1}{3}h)^2$$

$$= \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh \times \tfrac{1}{9}h^2 = \tfrac{1}{36}bh^3.$$

Problem 497.—Determine the moment of inertia of the area of a circle, in terms of its radius r, with respect to an axis coinciding with the diameter: (*a*) using Cartesian coordinates; (*b*) using polar coordinates.

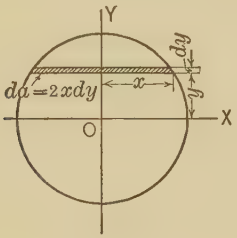


Fig. 357.

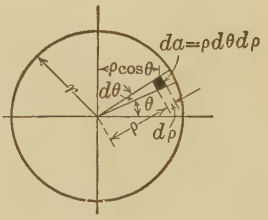


Fig. 358.

Solution. (*a*) **Cartesian Coordinates.**—The element of area will be selected as shown in Fig. 357. The moment of inertia of the circular area with respect to the diameter, then, is

$$I_x = \int y^2 da = \int y^2 2x dy$$

$$= 2\int_{-r}^{+r} y^2\sqrt{r^2 - y^2}dy = \tfrac{1}{4}\pi r^4.$$

(*b*) **Polar Coordinates.**—The element of area will be selected as shown in Fig. 358. Hence

$$I_x = \int y^2 da = \int_0^r \int_0^{2\pi} \rho^3 \sin^2\theta d\rho d\theta$$

$$= \frac{r^4}{4}\int_0^{2\pi} \sin^2\theta d\theta = \frac{r^4}{4} \times \pi = \tfrac{1}{4}\pi r^4 = \tfrac{1}{64}\pi d^4.$$

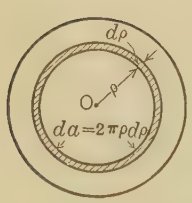


Fig. 359.

Problem 498.—Determine the polar moment of inertia of the area of a circle of radius r with respect to a centroidal axis: (*a*) by integration; (*b*) by the use of the theorem of Art. 168.

Solution. (*a*) **By Integration.**—By selecting the element of area as indicated in Fig. 359, the polar moment of inertia of the circular area is

$$J_z = \int \rho^2 da = \int_0^r \rho^2 2\pi\rho d\rho$$

$$= \tfrac{1}{2}\pi r^4 = \tfrac{1}{32}\pi d^4.$$

(*b*) **By Use of Theorem of Art. 168.**—Since I_x and I_y are each equal to $\frac{1}{4}\pi r^4$ (Prob. 497), the polar moment of inertia of the area of the circle is

$$J_z = I_x + I_y$$
$$= \tfrac{1}{4}\pi r^4 + \tfrac{1}{4}\pi r^4 = \tfrac{1}{2}\pi r^4.$$

PROBLEMS

499. Determine the moment of inertia of the area of a circle, with respect to an axis tangent to the circle, in terms of r, the radius of the circle.

500. Determine the polar moment of inertia of the area of a rectangle of base, b, and altitude, h, with respect to the centroidal axis. *Ans.* $\bar{J} = \frac{1}{12}bh(b^2 + h^2)$.

501. Find the moment of inertia and radius of gyration of a circular area 16 in. in diameter, with respect to a diameter.

502. Determine the moments of inertia of the area of an ellipse, the principal axes of which are $2b$ and $2h$, with respect to the principal axes.

Ans. $I_b = \frac{1}{4}\pi bh^3$. $I_h = \frac{1}{4}\pi hb^3$.

503. The base of a triangle is 8 in. and its altitude is 10 in. Find the moment of inertia and radius of gyration of the area of the triangle with respect to the base.

504. Find the polar moment of inertia and radius of gyration of the area of a square, each side of which is 15 in., with respect to an axis through one corner of the square.

505. Find the polar moment of inertia, with respect to a centroidal axis, of the area of an isosceles triangle having a base b and altitude h.

Ans. $J = \frac{1}{12}bh(\frac{1}{4}b^2 + \frac{1}{3}h^2)$.

172. Moments of Inertia of Composite Areas.—When a composite area can be divided into a number of simple areas, such as triangles, rectangles, and circles, for which the moments of inertia are known, the moment of inertia of the entire area may be obtained as the sum of the moments of inertia of the several areas. Likewise, the moment of inertia of the part of an area that remains after one or more simple areas are removed may be found by subtracting, from the moment of inertia of the given area, the sum of the moments of inertia of the several parts removed.

ILLUSTRATIVE PROBLEMS

Problem 506.—Locate the horizontal centroidal axis, XX, of the T-section shown in Fig. 360 and find the moment of inertia of the area with respect to this centroidal axis.

Solution. **First Method.**—The distance, $\bar{y}$, of the centroid of the area from the axis, X_1X_1, may be found from the equation

$$a\bar{y} = \Sigma(a'y_0).$$

Thus,

$$\bar{y} = \frac{12 \times 7 + 12 \times 3}{12 + 12}$$
$$= 5 \text{ in.}$$

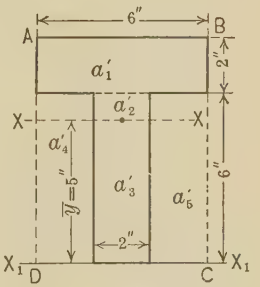


Fig. 360.

The moment of inertia with respect to the XX axis is the sum of the moments of inertia of the three parts a'_1, a'_2, and a'_3, with respect to that axis, thus

$$I_x = \tfrac{1}{12} \times 6 \times (2)^3 + 12 \times (2)^2 + \tfrac{1}{3} \times 2 \times (1)^3 + \tfrac{1}{3} \times 2 \times (5)^3$$

$$= 4 + 48 + 0.67 + 83.33 = 136 \text{ in.}^4$$

Second Method.—The moment of inertia of the T-section may also be determined as follows: First find the moment of inertia of the T-section with respect to the axis X_1X_1 by subtracting the moments of inertia of the parts a'_4 and a'_5 from the moment of inertia of the rectangular area $ABCD$, and then find $\bar{I}_x$ for the T-section by use of the parallel axis theorem. Thus, the moment of inertia, I_x, of the T-section with respect to the X_1X_1 axis is

$$I_x = \tfrac{1}{3} \times 6 \times (8)^3 - 2 \times \tfrac{1}{3} \times 2 \times (6)^3 = 736 \text{ in.}^4;$$

and

$$\bar{I}_x = I_x - ad^2 = 736 - 24 \times (5)^2 = 136 \text{ in.}^4$$

Problem 507.—Find the moment of inertia of the channel section shown in Fig. 361 with respect to the line XX. Find also the moment of inertia with respect to the parallel centroidal axis.

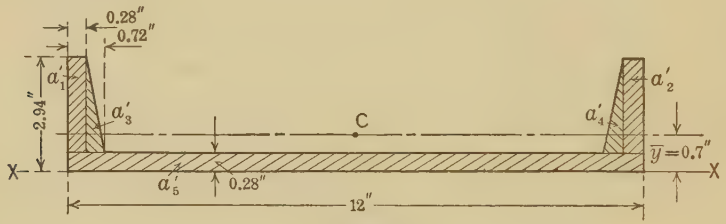


FIG. 361.

Solution.—The area may be divided into triangles and rectangles as shown in Fig. 361. The values used in the solution may be put in tabular form, as shown below, where a' denotes the area of any part, y_0 the distance of the centroid of the part from the line XX, I_0 the moment of inertia of the part with respect to its own centroidal axis parallel to XX, and I'_x the moment of inertia of the part with respect to the axis XX.

Part	a'	y_0	$a'y_0$	I_0	$a'y_0^2$	$I'_x = I_0 + ay_0^2$
a'_1	0.745	1.61	1.20	0.44	1.93	2.37
a'_2	0.745	1.61	1.20	0.44	1.93	2.37
a'_3	0.585	1.17	0.68	0.23	0.80	1.03
a'_4	0.585	1.17	0.68	0.23	0.80	1.03
a'_5	3.360	0.14	0.47	0.02	0.07	0.09
	6.02 in.^{2}		4.23 in.^{3}			6.89 in.^{4}

Thus the moment of inertia I_x of the area with respect to the XX-axis is

$$I_x = \Sigma I'_x = 6.89 \text{ in.}^4$$

Further, the total area is $a = \Sigma a' = 6.02$ in.^{2}, and the moment of the area with respect to the XX-axis is $\Sigma(a'y_0) = 4.23$ in.^{3} Hence, the distance, $\bar{y}$, of the centroid of the area from the XX-axis is

$$\bar{y} = \frac{\Sigma(a'y_0)}{a} = \frac{4.23}{6.02} = 0.70 \text{ in.}$$

Therefore, the moment of inertia with respect to a line through the centroid and parallel to XX is given by the equation

$$\bar{I}_x = I_x - ad^2$$
$$= 6.89 - 6.02 \times (0.70)^2 = 3.94 \text{ in.}^4$$

PROBLEMS

508. A wooden column is built up of four 2-in. by 8-in. planks as shown in Fig. 362. Find the moment of inertia of the cross-section with respect to the centroidal axis XX.

Ans. $I_x = 981$ in.^{4}

509. Find the moment of inertia of the angle section (Fig. 363) with respect to each of the centroidal axes parallel to the two legs of the angle.

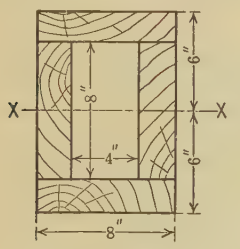


FIG. 362.

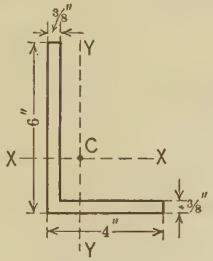


FIG. 363.

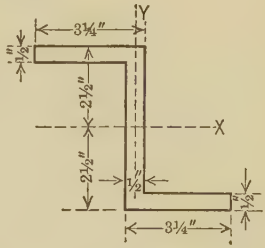


FIG. 364.

510. In Fig. 364 is shown the cross-section of a standard $3\frac{1}{4}$ in. by 5-in. Z-bar (fillets are neglected). Find the moments of inertia of the section with respect to the centroidal axes XX and YY.

511. In Fig. 365 is shown the cross-section of a standard 9-in. 21.8-lb. I-beam (fillets are neglected). Find the moments of inertia of the section with respect to the centroidal axes, XX and YY. *Ans.* $I_x = 84.9$ in.^{4}; $I_y = 5.16$ in.^{4}

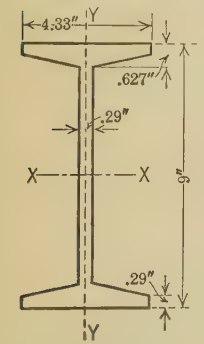


FIG. 365.

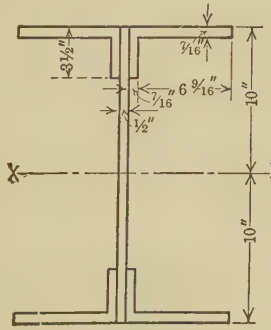


FIG. 366.

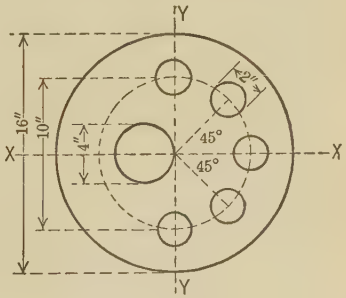


FIG. 367.

512. In Fig. 366 is shown a built-up section made of a $\frac{1}{2}$-in. by 20-in. plate and

four angles. Find the moment of inertia of the section with respect to the XX axis.

Ans. $I_x = 1850$ in.^{4}

513. In Fig. 367 is represented a 16-in. circular plate in which there are drilled five 2-in. holes and one 4-in. hole as shown. Find the moment of inertia of the area of the holes with respect to the XX axis and also with respect to the YY axis.

Ans. $I_x = 252$ in.^{4}

173. Approximate Method.—It is sometimes necessary to determine the moment of inertia of an area that has a boundary curve which cannot be defined by a mathematical equation. An approximate value of the moment of inertia of such an area may be found by the following method. For convenience, however, a simple area will be selected so that the approximate value of the moment of inertia as determined by this method may be compared with the exact value. Thus, let the moment of inertia of the area of a rectangle, with respect to an axis coinciding with its base, be found. The area may be divided into any convenient number of equal narrow strips parallel to the base, as shown in Fig. 368. (The narrower the strips the more closely will the result agree with the exact result.) Let the area be divided into ten such strips each 0.2 in. in width. The moment of inertia of the rectangle is equal to the sum of the moments of inertia of the strips. The moment of inertia of any particular strip with respect to the base of the rectangle is

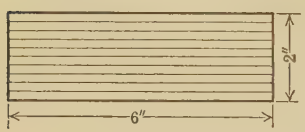


Fig. 368.

$$\tfrac{1}{12} \times 6 \times (\tfrac{1}{5})^3 + 6 \times \tfrac{1}{5} \times y^2,$$

where y is the distance of the centroid of the particular strip from the base. The first term is small and may be omitted without serious error. The moment of inertia of each strip then is approximately equal to the product of the area of the strip and the square of the distance of its centroid from the base. Hence, the moment of inertia of the rectangle is

$$I = \tfrac{6}{5}(.1^2 + .3^2 + .5^2 + .7^2 + .9^2 + 1.1^2 + 1.3^2 + 1.5^2 + 1.7^2 + 1.9^2)$$
$$= \tfrac{6}{5} \times 13.3 = 15.96 \text{ in.}^4$$

According to Prob. 495, the exact value is,

$$I = \tfrac{1}{3}bh^3 = \tfrac{1}{3} \times 6 \times 2^3 = 16 \text{ in.}^4$$

§ 3. Product of Inertia

174. Product of Inertia Defined.—If the moments of inertia of an area with respect to any two rectangular axes are known, the moment of inertia with respect to any other axis through the point of intersection

of the two axes may frequently be obtained most easily in terms of the moments of inertia of the area with respect to the two rectangular axes and an expression of the form $\int xydA$, in which dA is an element of the given area and x and y are the coordinates of the element with respect to the two rectangular axes. This expre sion is called the *product of inertia* of the area with respect to the axes and is denoted by P_{xy}. Hence the product of inertia of an area with respect to any two rectangular axes may be defined as the sum of the products obtained by multiplying each element of area by the product of the two coordinates of the element with respect to the two rectangular axes. That is,

$$P_{xy} = \int xydA. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (331)$$

The product of inertia of an area, like the moment of inertia of an area, is of four dimensions in length and is, therefore, expressed as inches (or feet, etc.) to the fourth power (in.^{4}, ft.^{4}, etc.). Unlike moment of inertia, however, the product of inertia of an area is not always positive, but may be negative or may be zero, since either x or y may be negative and hence their product may be negative, and the sum of the products may be equal to zero.

Axes of Symmetry.—The product of inertia of an area with respect to two rectangular axes is zero if either one of the axes is an axis of symmetry. This follows from the fact that, for each product $xyda$ for an element on one side of the axis of symmetry, there is an equal product of opposite sign for the corresponding element on the other side of the axis, and hence the expression $\int xyda$ equals zero.

ILLUSTRATIVE PROBLEMS

Problem 514.—Find the product of inertia of the area of the triangle, shown in Fig. 369, with respect to the x- and y-axes.

Solution.—The relation between x and y is given by the equation $y = \frac{2}{3}x$. Hence

$$P_{xy} = \int xyda = \int_0^6 \int_0^{\frac{2}{3}x} xydxdy$$

$$= \frac{2}{9}\int_0^6 x^3dx = 72 \text{ in.}^4$$

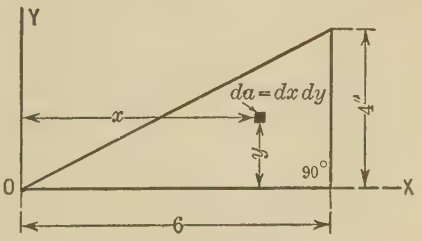


FIG. 369.

Problem 515.—Find the product of inertia of the area of the quadrant of a circular area (Fig. 370) with respect to the x and y axes, in terms of its radius r.

First Method.—Let the elementary area be selected as shown in Fig. 370(*a*) and expressed in terms of rectangular coordinates; then

$$P_{xy} = \int_0^r \int_0^{\sqrt{r^2-x^2}} xy\,dx\,dy = \int_0^r x\left[\frac{y^2}{2}\right]_0^{\sqrt{r^2-x^2}} dx$$

$$= \int_0^r x \cdot \frac{r^2 - x^2}{2}\,dx = \tfrac{1}{2}\int_0^r (r^2x - x^3)dx = \tfrac{1}{2}\left[\frac{r^2x^2}{2} - \frac{x^4}{4}\right]_0^r = \tfrac{1}{8}r^4.$$

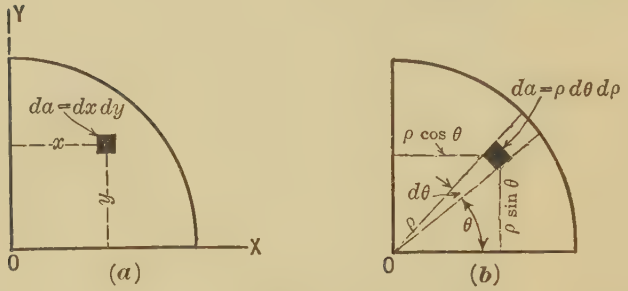


FIG. 370.

Second Method.—Let the elementary area be selected as shown in Fig. 370(*b*) and be expressed in terms of polar coordinates; then

$$P_{xy} = \int xy\,da = \int_0^r \int_0^{\pi/2} \rho\cos\theta \cdot \rho\sin\theta \cdot \rho\,d\rho\,d\theta$$

$$= \int_0^r \int_0^{\pi/2} \rho^3 d\rho \cdot \tfrac{1}{2}\sin 2\theta\,d\theta = \frac{1}{2}\frac{r^4}{4}\int_0^{\pi/2} \sin 2\theta\,d\theta$$

$$= \frac{r^4}{8}\left[-\cos 2\theta \cdot \frac{1}{2}\right]_0^{\theta=\pi/2} = -\frac{r^4}{16}(-2) = \tfrac{1}{8}r^4.$$

PROBLEMS

516. Find the product of inertia of the area of a rectangle having a base b and an altitude h, with respect to two adjacent sides. *Ans.* $P_{xy} = \pm\,\frac{1}{4}b^2h^2$.

517. Find the product of inertia of a right-angle triangle (Fig. 371) with respect to the x- and y-axes shown, the x-axis being a centroidal axis.

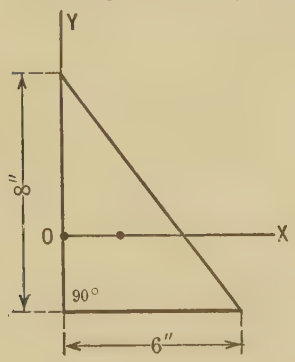


FIG. 371.

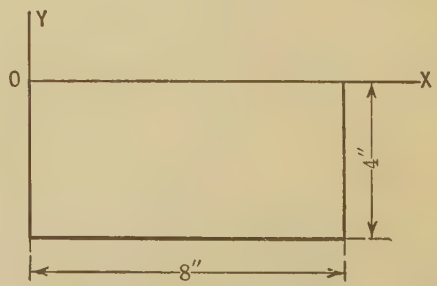


FIG. 372.

518. Find the product of inertia of the rectangular area with respect to the x- and y-axes as shown in Fig. 372.

519. Find the product of inertia, with respect to the coordinate axes, of the area bounded by the parabola $y^2 = hx$, the line $x = b$, and the x-axis.

Ans. $P_{xy} = \frac{1}{6}hb^3$.

175. Parallel Axis Theorem for Products of Inertia.—When the product of inertia of an area is known for any pair of rectangular axes passing through the centroid of the area, the product of inertia of the area with respect to any parallel set of axes may be determined without integrating. Thus, in Fig. 373, $X'X'$ and $Y'Y'$ are axes through the centroid, C, of the area; XX and YY are parallel axes passing through the point O. The coordinates of C with respect to XX and YY are denoted by $\bar{x}$ and $\bar{y}$. If the product of inertia of the area with respect to XX and YY be denoted by P_{xy} and the product of inertia with respect to $X'X'$ and $Y'Y'$ be denoted by $\overline{P}_{xy}$, then, by definition,

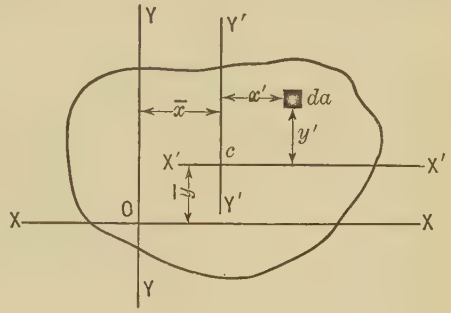


Fig. 373.

$$P_{xy} = \int (x' + \bar{x})(y' + \bar{y})da$$

$$= \int x'y'da + \bar{x}\bar{y}\int da + \bar{y}\int x'da + \bar{x}\int y'da.$$

Since each of the last two integrals is the first moment of the area with respect to a centroidal axis, each integral is equal to zero. The equation then becomes,

$$P_{xy} = \overline{P}_{xy} + a\bar{x}\bar{y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (332)$$

That is, the product of inertia of any area with respect to any pair of rectangular axes in its plane is equal to the product of inertia of the area with respect to a pair of parallel centroidal axes plus the product of the area and the coordinates of the centroid of the area with respect to the given pair of axes.

ILLUSTRATIVE PROBLEM

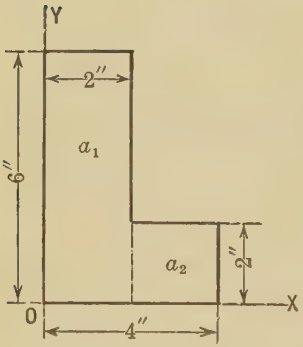


Fig. 374.

Problem 520.—Find the product of inertia of the area shown in Fig. 374 with respect to the x- and y-axes.

Solution.—The area may be divided into rectangles a_1 and a_2 as shown. Using the formula $P_{xy} = \overline{P}_{xy} + a \cdot \overline{xy}$, we have, for the area a_1,

$$P_{xy} = 0 + 12 \times 1 \times 3 = 36 \text{ in.}^4$$

and for area a_2,

$$P_{xy} = 0 + 4 \times 3 \times 1 = 12 \text{ in.}^4$$

Hence, for the entire area,

$$P_{xy} = 36 + 12 = 48 \text{ in.}^4$$

PROBLEMS

521. Find the product of inertia of the area shown in Fig. 375 with respect to the x- and y-axes. *Ans.* $P_{xy} = 561 \text{ in.}^4$

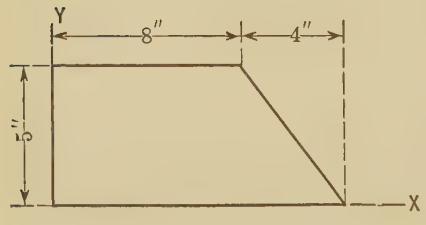


Fig. 375.

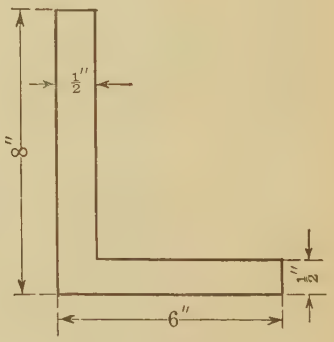


Fig. 376.

522. Locate the centroid of the angle section shown in Fig. 376 and determine the product of inertia with respect to centroidal axes parallel to the two legs of the angle.

176. Relation between Moments of Inertia and Products of Inertia with Respect to Two Sets of Rectangular Axes through the Same Point in the Area.—Let I_x, I_y, P_{xy} denote the moments of inertia and product of inertia of the area in Fig. 377 with respect to the axes OX and OY. Let it be required to find the moments of inertia, I'_x and I'_y, and the product of inertia, P'_{xy}, with respect to the axes OX' and OY' which are inclined an angle θ with the axes OX and

OY. The moment of inertia of the area in Fig. 377 with respect to the axis OX' is expressed by the equation

$$I'_x = \int y'^2 da = \int (y\cos\theta - x\sin\theta)^2 da$$

$$= \cos^2\theta \int y^2 da + \sin^2\theta \int x^2 da - 2\sin\theta\cos\theta \int xy da$$

$$= I_x \cos^2\theta + I_y \sin^2\theta - 2P_{xy}\sin\theta\cos\theta. \quad . \quad . \quad . \quad . \quad (333)$$

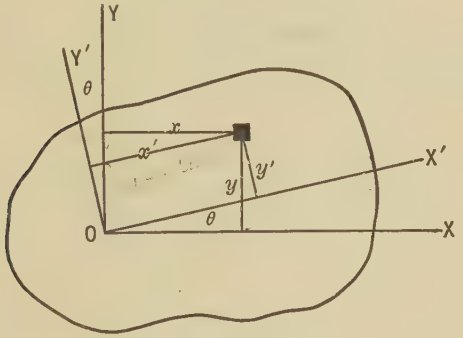


Fig. 377.

In a similar manner the following equation may be derived.

$$I'_y = I_x \sin^2\theta + I_y \cos^2\theta + 2P_{xy}\sin\theta\cos\theta. \quad . \quad . \quad (334)$$

Likewise

$$P'_{xy} = \int x'y' da = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta) da$$

$$= (\cos^2\theta - \sin^2\theta) \int xy da + (\cos\theta\sin\theta) \int (y^2 - x^2) da$$

$$= P_{xy}\cos 2\theta + \tfrac{1}{2}(I_x - I_y)\sin 2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (335)$$

Thus from eqs. (333), (334), and (335), the moments of inertia and product of inertia of an area with respect to any set of rectangular axes may be found, without integrating, in terms of the moments of inertia and product of inertia with respect to a given set of rectangular axes passing through the same origin.

By adding eqs. (333) and (334), the following important equation is found:

$$I'_x + I'_y = I_x + I_y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (336)$$

That is, the sum of the moments of inertia of an area with respect to all pairs of rectangular axes having a common point of intersection is constant. It should be noted also that each side of eq. (336) is equal to

the polar moment of inertia of the area with respect to an axis intersecting the area at the common point.

Axes for which the Product of Inertia is Zero.—It may be shown that through any point in an area one set of rectangular axes may be drawn for which the product of inertia is zero. Thus, in eq. (335), if P'_{xy} is made equal to zero, we obtain the equation

$$\tan 2\theta = -\frac{P_{xy}}{\frac{1}{2}(I_x - I_y)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (337)$$

and hence, when θ (Fig. 377) has a value given by this equation, the product of inertia with respect to the OX' and OY' axes is zero; in the next article it will be shown that the axes for which the product of inertia is zero are principal axes.

177. Principal Axes.—In the analysis of many engineering problems the moment of inertia of an area must be found with respect to a certain axis called a principal axis. A principal axis of inertia of an area, for a given point in the area, is an axis about which the moment of inertia of the area is either greater or less than for any other axis passing through the given point. It can be proved that through any point in an area two rectangular axes can be drawn for which the moments of inertia of the area are greater and less, respectively, than for any other axes through the point. There are then two principal axes of inertia of an area for any point in the area. Further, it can be shown that axes for which the product of inertia is zero are principal axes. And since the product of inertia of an area is zero for axes of symmetry, it follows that axes of symmetry are principal axes. The above statements may be demonstrated as follows:

The direction of the principal axes may be determined from eq. (333) of Art. 176 which may be written in the form

$$I'_x = I_x \frac{1 + \cos 2\theta}{2} + I_y \frac{1 - \cos 2\theta}{2} - P_{xy} \sin 2\theta$$

$$= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - P_{xy} \sin 2\theta. \quad . \quad . \quad . \quad (338)$$

The value of θ which will make I'_x have a maximum or a minimum value may be found by equating to zero the first derivative of I'_x with respect to θ. Thus,

$$\frac{dI'_x}{d\theta} = \sin 2\theta(I_y - I_x) - 2P_{xy} \cos 2\theta = 0$$

$$\tan 2\theta = \frac{2P_{xy}}{I_y - I_x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (339)$$

which is the same as eq. (337). From this equation two values of 2θ are obtained which differ by 180°, the corresponding values of θ differing by 90°. For one value of θ the value of I'_x will be a maximum, and for the other, a minimum. If $P_{xy} = 0$ (which will always be the case if either the x- or y-axis is an axis of symmetry), the value of θ is zero, and hence axes of symmetry are principal axes.

ILLUSTRATIVE PROBLEM

Problem 523.—Find the moments of inertia of the angle section, shown in Fig. 378 with respect to principal axes passing through the centroid.

Solution.—The steps in the solution will be made as follows:

(1) The centroid of the area will be located, that is, $\bar{x}$ and $\bar{y}$ will be found.

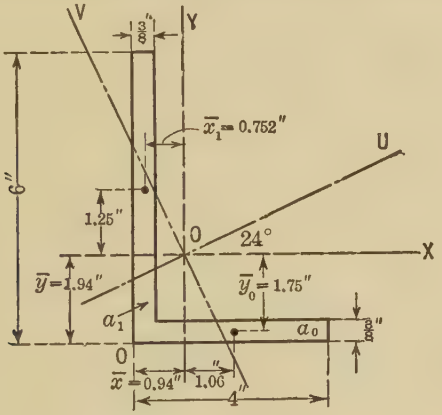


Fig. 378.

(2) The moments of inertia and the product of inertia ($\bar{I}_x$ and $\bar{I}_y$, and $\bar{P}_{xy}$) with respect to the centroidal x- and y-axes will then be found by the methods discussed in Art. 172.

(3) The directions of the principal axes will then be found by use of the equations of Art. 177.

(4) The moment of inertia with respect to each of the principal axes, u and v, will then be found by means of eqs. (333) and (334).

$$\bar{x} = \frac{4 \times \frac{3}{8} \times 2 + 5\frac{5}{8} \times \frac{3}{8} \times \frac{3}{16}}{4 \times \frac{3}{8} + 5\frac{5}{8} \times \frac{3}{8}} = \frac{3.396}{3.61} = 0.94 \text{ in.}$$

$$\bar{y} = \frac{4 \times \frac{3}{8} \times \frac{3}{16} + 5\frac{5}{8} \times \frac{3}{8} \times 3\frac{3}{16}}{4 \times \frac{3}{8} + 5\frac{5}{8} \times \frac{3}{8}} = \frac{7.01}{3.61} = 1.94 \text{ in.}$$

$$\bar{I}_x = \tfrac{1}{12} \times \tfrac{3}{8} \times (5\tfrac{5}{8})^3 + 5\tfrac{5}{8} \times \tfrac{3}{8} \times (1\tfrac{1}{4})^2 + \tfrac{1}{12} \times 4 \times (\tfrac{3}{8})^3 + 4 \times \tfrac{3}{8} \times (1\tfrac{3}{4})^2$$

$$= 5.57 + 3.30 + 0.02 + 4.59 = 13.48 \text{ in.}^4$$

$$\bar{I}_y = \tfrac{1}{12} \times 5\tfrac{5}{8} \times (\tfrac{3}{8})^3 + 5\tfrac{5}{8} \times \tfrac{3}{8} \times (\tfrac{3}{4})^2 + \tfrac{1}{12} \times \tfrac{3}{8} \times 4^3 + 4 \times \tfrac{3}{8} \times (1\tfrac{1}{16})^2$$
$$= 0.02 + 1.19 + 2.00 + 1.69 = 4.90 \text{ in.}^4$$

The value of $\bar{P}_{xy}$ may be found by means of formula (332). Thus,

$$\bar{P}_{xy} = \bar{P}_{0xy} + a_0\bar{x}_0\bar{y}_0 + \bar{P}_{1xy} + a_1\bar{x}_1\bar{y}_1$$
$$= 4 \times \tfrac{3}{8} \times 1.06 \times (-1.75) + 5.62 \times \tfrac{3}{8} \times 1.25 \times (-0.752)$$
$$= -4.76 \text{ in.}^4$$

The directions of the principal axes are found from formula (339). Thus,

$$\tan 2\theta = \frac{2 \times (-4.76)}{4.90 - 13.48} = 1.11$$

$$2\theta = 48^\circ \quad \text{or} \quad 228^\circ; \quad \theta = 24^\circ \quad \text{or} \quad 114^\circ.$$

From formula (333) the moment of inertia with respect to the axis making an angle of 24° with OX (denoted by u) is

$$I_u = 13.48 \cos^2 24^\circ + 4.90 \sin^2 24^\circ - 2(-4.76) \sin 24^\circ \cos 24^\circ$$
$$= 11.23 + 0.81 + 3.53 = 15.59 \text{ in.}^4$$

Using $\theta = 114^\circ$ and denoting the corresponding axis by v, we have

$$I_v = 13.48 \cos^2 114^\circ + 4.90 \sin^2 114^\circ - 2(-4.76) \sin 114^\circ \cos 114^\circ$$
$$= 2.23 + 4.08 - 3.53 = 2.78 \text{ in.}^4$$

PROBLEMS

524. In the Z-section shown in Fig. 379, $\bar{I}_x = 25.32$ in.^{4} and $\bar{I}_y = 9.11$ in.^{4} Find the principal moments of inertia. *Ans.* 3.10 in.^{4}; 31.3 in.^{4}

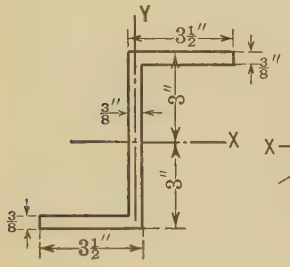


Fig. 379.

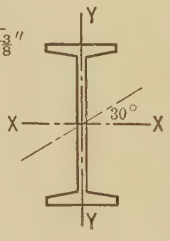


Fig. 380.

525. Show that the moment of inertia of the area of a square is constant for all axes in the plane of the area which pass through the center.

526. Fig. 380 represents the cross-section of a standard 10-in. 25-lb. I-beam. $\bar{I}_x = 122.1$ in.^{4}, $\bar{I}_y = 6.89$ in.^{4}, and $a = 7.38$ in.^{2} Find the moment of inertia and radius of gyration of the section with respect to a line making an angle of 30° with the x-axis.

Ans. $I = 93.3$ in.^{4}

178. Graphical Solution.—Let the moments of inertia, I_x and I_y, and the product of inertia P_{xy} of an area with respect to a given set of rectangular axes, OX and OY, in the area be known (Fig. 377); and let it be required to find the moments of inertia, I'_x and I'_y, and the product of inertia, P'_{xy}, with respect to another set of rectangular axes, OX' and OY', passing through the same point in the area, in terms of

the known values I_x, I_y, P_{xy} and the angle θ between the axes OX and OX'.

Eqs. (333), (334), and (335) give the algebraic expressions for I'_x, I'_y, and P'_{xy}, and it will now be shown that the equations may be solved graphically as follows.

Mohr-Land Construction.—Fig. 381 (a) is constructed as follows:

1. On OY lay off $OC = I_x$ and $CD = I_y$; $I_x > I_y$.
2. Lay off $CT = P_{xy}$ parallel to OX (+ to right, − to left).
3. On OD as a diameter draw a circle; M is the center.
4. Draw OX' making the given angle θ with OX, cutting the circle at E.
5. Draw the diameter EH and draw TG perpendicular to EH.

Then,

$$TG = P'_{xy},\ EG = I'_x \quad \text{and} \quad HG = I'_y.$$

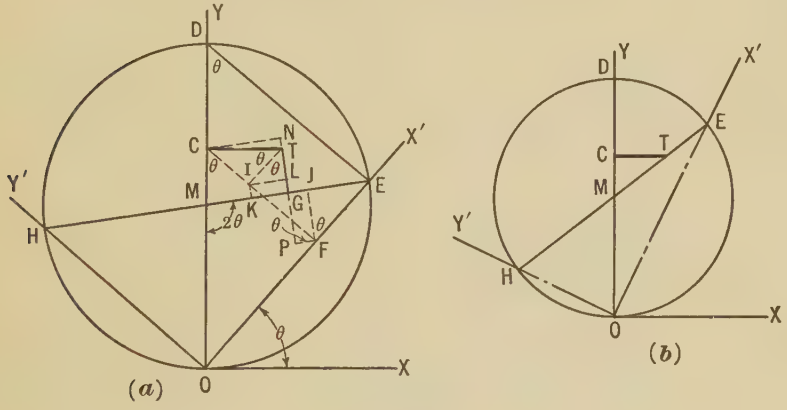


FIG. 381.

Proof.—Draw CF and $DE \perp OE$; $TI \perp CF$; FJ and $IK \perp HE$; IL, CN, and $FP \perp TG$.

Then

$$\angle CFP = \angle ODE = \angle OCF = \angle ITC = \angle ITL = \angle JFE = \theta$$

$$\angle OCN = \angle OME = 2\theta \text{ and } \angle CTN = 180 - 2\theta.$$

$$GE = JK + JE - GK.$$

Now,

$$JK = FI \sin \angle CFJ = FI \cos \angle CFP = (CF - IC) \cos \theta$$

$$= (I_x \cos \theta - P_{xy} \sin \theta) \cos \theta$$

$$JE = FE \sin \angle JFE = I_y \sin^2 \theta$$

$$GK = IL = CT \cos \theta \sin \theta = P_{xy} \sin \theta \cos \theta.$$

Therefore,

$$GE = I_x \cos^2\theta + I_y \sin^2\theta - 2P_{xy} \sin\theta \cos\theta = I'_x.$$

Also,

$$GH = HE - GE = OD - GE = I_x + I_y - I'_x = I'_y$$

(see eq. 336)

$$TG = NP - NT - PG$$

$$NP = CF \sin \angle FCN = I_x \cos\theta \sin\theta$$

$$NT = CT \cos \angle CTN = CT \cos(180 - 2\theta) = - CT \cos 2\theta$$

$$= - P_{xy} \cos 2\theta$$

$$PG = FJ = FE \cos \angle JFE = I_y \cos\theta \sin\theta$$

$$TG = \tfrac{1}{2} I_x \sin 2\theta + P_{xy} \cos 2\theta - \tfrac{1}{2} I_y \sin 2\theta$$

$$= \tfrac{1}{2}(I_x - I_y) \sin 2\theta + P_{xy} \cos 2\theta = P'_{xy}.$$

Principal Axes and Principal Moments of Inertia.—Draw a diameter HE through T (Fig. 381*b*), thus making $P'_{xy} = 0$ (see Art. 177). Then, OE and OH are the principal axes, and TE and TH are principal moments of inertia.

PROBLEMS

527. Solve Prob. 524 graphically.

528. Solve Prob. 526 graphically.

§ 4. Values of Z for Use in Curved-beam Formula

179. Values of Z found by Integration.—By definition we have (see Art. 142).

$$Z = -\frac{1}{a}\int \frac{y}{R+y}\,da. \quad \ldots\ldots \quad (340)$$

For sections of simple form, such as rectangular and circular sections, the value of Z may easily be found by direct integration. The value of Z for trapezoidal sections, I-sections, etc., may also be found by direct integration, but the expressions are rather clumsy to use; it is often convenient therefore, and sometimes necessary, to obtain a value of Z by a graphical method.

By writing the dimensional equation for Z it becomes evident that Z is an abstract number, and hence its value will be the same regardless of the unit of length used for expressing the value of y, R and a, provided, of course, that the same unit of length is used for each of these three quantities.

Rectangular Area.—Let Fig. 382 represent a normal section of a curved beam having a rectangular section of width b and depth $2c$, the distance from the centroidal axis of the section to the axis of curvature being R. Then

$$Z = -\frac{1}{2bc}\int_{-c}^{+c} \frac{y}{R+y}\, b\,dy = -\frac{1}{2c}\int_{-c}^{+c} \frac{y}{R+y}\, dy. \quad . \quad (341)$$

First Method.—If $\dfrac{1}{R+y} = (R+y)^{-1} = \dfrac{1}{R}\left(1+\dfrac{y}{R}\right)^{-1}$ is expanded

in a converging series by the binomial theorem we have

$$Z = -\frac{1}{2c}\int_{-c}^{+c}\left(\frac{y}{R} - \frac{y^2}{R^2} + \frac{y^3}{R^3} - \frac{y^4}{R^4} + \dots\right)dy. \quad . \quad . \quad (342)$$

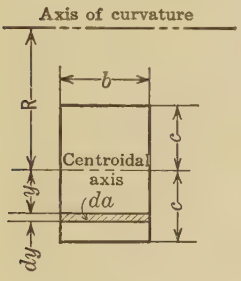


Fig. 382.

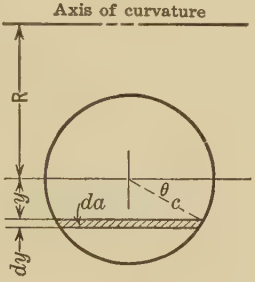


Fig. 383.

By integrating and substituting limits, the following expression is found

$$Z = \frac{1}{3}\left(\frac{c}{R}\right)^2 + \frac{1}{5}\left(\frac{c}{R}\right)^4 + \frac{1}{7}\left(\frac{c}{R}\right)^6 + \dots \quad . \quad . \quad . \quad . \quad . \quad (343)$$

Second Method.—Eq. (341) may be integrated directly; thus

$$Z = -\frac{1}{2c}\int_{-c}^{+c}\left(1 - \frac{R}{R+y}\right)dy$$

$$Z = -\frac{1}{2c}\Big[y - R\log_e(R+y)\Big]_{-c}^{c}$$

Hence

$$Z = -1 + \frac{R}{2c}\log_e\left(\frac{R+c}{R-c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (344)$$

Circular Area. First Method.—Let c denote the radius of the circular area; then as indicated in Fig. 383:

$$da = 2c \cos \theta dy, \quad y = c \sin \theta.$$

Hence

$$dy = c \cos \theta d\theta.$$

Therefore,

$$Z = -\frac{1}{a}\int \frac{y}{R+y}\,da = -\frac{1}{\pi c^2}\int_{-\pi/2}^{+\pi/2} \frac{c \sin \theta}{R + c \sin \theta}\, 2c^2 \cos^2 \theta d\theta$$

$$= -\frac{2c}{\pi}\int_{-\pi/2}^{+\pi/2} \frac{\cos^2 \theta \sin \theta d\theta}{R + c \sin \theta}. \quad \ldots \ldots \ldots \quad (345)$$

If $\dfrac{1}{R + c \sin \theta}$ is expanded in a converging series and if each term then under the integral is integrated separately the final result is

$$Z = \frac{1}{4}\left(\frac{c}{R}\right)^2 + \frac{1}{8}\left(\frac{c}{R}\right)^4 + \frac{5}{64}\left(\frac{c}{R}\right)^6 + \frac{7}{128}\left(\frac{c}{R}\right)^8 + \frac{21}{512}\left(\frac{c}{R}\right)^{10} + . \quad (346)$$

Second Method.—Eq. (345) may also be integrated as follows:

$$Z = -\frac{2c}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\cos^2 \theta \sin \theta d\theta}{R + c \sin \theta}$$

$$= -\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sin \theta - \sin^3 \theta}{\sin \theta + m}\, d\theta, \text{ where } m = \frac{R}{c}$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \left[\sin^2 \theta - m \sin \theta + (m^2 - 1) - \frac{m(m^2-1)}{\sin \theta + m}\right] d\theta$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} + (m^2 - 1)\pi\right] - 2m\sqrt{m^2 - 1} = 2m^2 - 2m\sqrt{m^2 - 1} - 1$$

$$= -1 + 2\left(\frac{R}{c}\right)^2 - 2\left(\frac{R}{c}\right)\sqrt{\left(\frac{R}{c}\right)^2 - 1} \quad \ldots \ldots \ldots \quad (347)$$

APPENDIX II

TABLES OF PROPERTIES OF MATERIALS

By H. F. Moore [1]

180. Use of Tables.—In this chapter are grouped, for convenience, tables giving average values of the properties of the more common structural materials. In using these tables it should be remembered that the values given are average values, and that the values of the properties of any one material may vary considerably from those given in these tables. Therefore, the values given should not be used blindly; the uniformity and general quality of the material should be considered in addition to the properties. If possible, structures and machines in which the material has been used should be studied. When average values of the properties of a material are the only information available the values used in design should err on the side of safety.

[1] Research Professor of Engineering Materials, University of Illinois.

TABLE I

Average Values for Static Strength, Stiffness and Ductility of Iron and Steel

The values given in this table are based on test results from various laboratories.

Material	Strength in Tension Lb. per Sq. In.		Strength in Compression Lb. per Sq. In.		Strength in Shear Lb. per Sq. In.		Modulus of Elasticity Lb. per Sq. In.		Elongation in 2 Inches Per Cent	Brinell Hardness Number
	Proportional Elastic Limit	Ultimate	Proportional Elastic Limit	Ultimate	Proportional Elastic Limit	Ultimate	Tension	Shear		
Gray cast iron	(a)	20,000	26,000	75,000	(b)	(b)	15,000,000	6,000,000	Slight	(c)
"Semi steel"	14,000	35,000	50,000	130,000	(b)	(b)	(c)	(c)	Slight	(c)
Malleable cast iron	20,000	50,000	20,000	(d)	10,000	(c)	23,000,000	10,500,000	10.0	80
Ingot iron, annealed (Commercial pure iron)	16,000	42,000	19,000	(d)	12,000	30,000	30,000,000	12,000,000	48.0	69
Commercial Wrought iron	30,000	50,000	30,000	(d)	18,000	35,000	27,000,000	10,000,000	35.0	100
Steel, 0.10% carbon, hot-rolled	30,000	50,000	30,000	(d)	18,000	35,000	30,000,000	12,000,000	45.0	100
0.20% carbon, hot-rolled	35,000	60,000	35,000	(d)	21,000	45,000	30,000,000	12,000,000	35.0	120
cold-rolled or cold-drawn	60,000	80,000	60,000	(d)	36,000	60,000	30,000,000	12,000,000	18.0	160
annealed castings	30,000	60,000	30,000	(d)	18,000	45,000	30,000,000	12,000,000	30.0	120
Steel, 0.40% carbon, hot-rolled	40,000	70,000	40,000	(d)	24,000	55,000	30,000,000	12,000,000	25.0	135
heat-treated for fine grain	60,000	90,000	60,000	(d)	36,000	75,000	30,000,000	12,000,000	25.0	210
annealed castings	32,000	65,000	30,000	(d)	18,000	55,000	30,000,000	12,000,000	15.0	130
Steel, 0.60% carbon, hot-rolled	60,000	100,000	60,000	(d)	36,000	80,000	30,000,000	12,000,000	15.0	200
heat-treated for fine grain	75,000	120,000	75,000	(d)	45,000	100,000	30,000,000	12,000,000	15.0	235
Steel, 0.80% carbon, hot-rolled	70,000	120,000	70,000	(d)	42,000	105,000	30,000,000	12,000,000	10.0	240
heat-treated for fine grain	90,000	150,000	90,000	(d)	54,000	125,000	30,000,000	12,000,000	9.0	300
oil-quenched	120,000	180,000	120,000	(d)	72,000	150,000	30,000,000	12,000,000	2.0	370
Steel, 1.00% carbon, hot-rolled	80,000	135,000	80,000	(d)	48,000	115,000	30,000,000	12,000,000	10.0	260
heat-treated for fine grain	120,000	175,000	120,000	(d)	72,000	148,000	30,000,000	12,000,000	9.0	350
oil-quenched	135,000	220,000	135,000	(d)	80,000	185,000	30,000,000	12,000,000	1.0	430
Structural Nickel steel, 3.5% nickel, hot-rolled	55,000	95,000	55,000	(d)	33,000	80,000	30,000,000	12,000,000	25.0	190
Nickel steel, 3.5% nickel, 0.40% carbon, maximum hardness for machineability	150,000	170,000	150,000	(d)	90,000	140,000	30,000,000	12,000,000	9.0	350
Oil-quenched	160,000	285,000	160,000	(c)	96,000	(c)	30,000,000	12,000,000	5.0	500
Other alloy steels for machine parts, see special note below.										
Silico-manganese steel, 1.95% Si., 0.70% Mn Spring tempered	130,000	174,000	130,000	(c)	78,000	115,000	30,000,000	12,000,000	14.0	360
Nickel Steel, 15% nickel	140,000	170,000	(c)	(c)	(c)	(c)	(c)	(c)	Slight	(c)
Wire (not annealed) (e):										
Iron or soft steel	70,000	85,000	(f)	(f)	(f)	(f)	30,000,000	(f)	(c)	(c)
High-carbon steel	150,000	200,000	(f)	(f)	(f)	(f)	30,000,000	(f)	(c)	(c)

Special Note on Alloy Steels: The strength properties of the various alloy steels used for machine parts—such as chrome-nickel steel, chrome-vanadium steel, molybdenum steel, etc.—depend on the heat-treatment used rather than on the special ingredients present. The range of strength values for any of the above-named steels is about the same as for nickel steel.

(a) No well-defined proportional elastic limit. (b) Greater than the tensile strength: under pure shear cast iron fails by tension on an oblique plane.
(c) Data lacking. (d) For ductile material the yield-point, which is slightly greater than the proportional elastic limit, is the practical ultimate in compression.
(e) For annealed wire the values are about the same as for the corresponding grade of rolled steel. (f) Wire can offer resistance only in tension.

TABLE II

Average Values for Static Strength and Ductility of Various Non-ferrous Metals and Alloys

Values given are based on test data from various testing laboratories

Metal or Alloy	Approximate Composition Per Cent	Weight Lb. per Cu. In.	Strength in Tension (*a*)		Elongation in 2 In. Per Cent
			Proportional Elastic Limit Lb. per Sq. In.	Ultimate Tensile Strength Lb. per Sq. In.	
Copper, annealed	Copper, 100	0.320	3,200	32,000	56
cold-drawn			38,000	50,000	8
Zinc, cast	Zinc, 100	0.253	.....	9,000	Slight
rolled			5,000	24,000	50
Tin, cast	Tin, 100	0.260	.....	4,000 (*b*)	....
rolled			.....	5,000 (*c*)	....
Lead, cast	Lead, 100	0.410	.....	1,700 (*d*)	....
rolled			.....	3,300 (*e*)	....
Common brass, cast	Copper, 60; zinc, 40	0.290	20,000	55,000	20
rolled			25,000	65,000	30
Phosphor bronze, cast	Copper, 95; tin, 4.9; phosphorus trace	0.32	10,000	32,000	7
rolled			40,000	65,000	30
hard-drawn			.....	105,000	5
spring wire					
Aluminum bronze, cast	Copper, 90; aluminum, 10	0.27	25,000	60,000	25
rolled			30,000	70,000	30
Monel metal, cast	Nickel, 67: copper, 28; iron + carbon + manganese + silicon, 5	0.32	37,000	72,000 (*f*)	34
rolled			50,000	85,000	42

General Note: A large number of special alloys of copper, tin, zinc, and other metals are in use whose physical properties differ somewhat from those tabulated here. The physical properties tabulated above are only general averages, and they may be materially modified by heat treatment, and very markedly modified by cold-rolling or cold drawing.

(*a*) Where no values for compressive strength are noted the ultimate in compression may be assumed as having the same value as the proportional limit in tension. The strength in shear may be taken as 60 per cent of the strength in tension.

(*b*) Ultimate in compression, 6400 lb. per sq. in.

(*c*) Modulus of elasticity in tension, 4,000,000 lb. per sq. in.

(*d*) Modulus of elasticity in tension, 700,000 lb. per sq. in.

(*e*) Modulus of elasticity in tension, 1,000,000 lb. per sq. in.

(*f*) Modulus of elasticity in tension, 22,000,000 lb. per sq. in.

TABLE III

Average Values for Static Strength and Ductility of Light Metal Alloys

The values given in this table are based on data from various testing laboratories, especially from the laboratory of the U. S. Air Service at McCook Field, Dayton, Ohio.

Metal	Weight Lb. per Cu. In.	Tensile strength* Lb. per Sq. In.		Elongation In 2 In. Per Cent
		Proportional Elastic Limit	Ultimate	
Commercial aluminum, 99% pure:				
cast	0.093	9,000	13,000	20
rolled and annealed	0.097	8,500	13,500	23
hard-drawn	0.097	20,000	30,000	4
hard-drawn wire	0.097	30,000	40,000	4
Aluminum 96%, copper 4%:				
cast	0.104	11,500	19,500	12
hard-drawn	0.104	35,000	41,000	5
Duralumin; aluminum 96%, magnesium 1%, copper 2.9%, traces of iron, silicon, and manganese:				
annealed	0.102	6,800	25,200	18
tempered	0.102	18,500	51,200	29
Electron metal, magnesium 95%, zinc 4.4%, small quantities of copper, iron, silicon, and aluminum:				
rolled	0.065	6,800	25,200	25

* The compressive strength of metals here tabulated may be safely taken as equal to the proportional elastic limit in tension. The strength in shear may safely be taken as 60 per cent of the tensile strength.

TABLE IV

General Properties and Uses of Wood

Wood	General Properties	Used for
Hard Woods:		
White Oak	Strong and tough, close grained, splits with difficulty, heavy, must be carefully seasoned to avoid checking.	Ties, vehicle and furniture making, interior finish, framing where great strength is of first consequence.
Red Oak	A softer, weaker, and more porous kind of oak than white oak.	Ties, interior finish, furniture.
Hickory	The strongest, toughest, and heaviest of American woods. Susceptible to the attacks of boring insects.	Vehicle and agricultural implement manufacture.
Maple	Rather coarse grained, but heavy and strong. It takes a fine polish.	Flooring, furniture, interior finish.
Elm	Strong and tough, but difficult to split and shape. It warps badly.	Vehicle and ship building.
Ash	Strong, but not very tough.	Interior finish.
Soft Woods:		
Spruce	Light, soft, straight grained, resistant to decay.	Framing timbers, piles, under water construction.
Douglas Fir (Oregon Fir)	Strong, though rather variable in quality, durable.	All kinds of construction.
White Pine	Light, soft, straight grained, not very strong.	Pattern making, interior finish.
Norway Pine (Red Pine)	Hard, light, coarse grained.	All kinds of construction.
Western Pine	Trade name for a number of kinds of wood with general characteristics somewhat like Norway pine.	General construction work.
Yellow Pine	The strongest and toughest of the soft woods. Heavy, decays in contact with soil.	Heavy framing timbers, flooring,
Hemlock	Light, brittle, splits easily.	Cheap framing lumber, crates and boxes.
Tamarack (Larch)	Strong, heavy, durable	Ties, sills, posts and poles, ship lumber.
Cedar	Light, durable, but not strong.	Water tanks, shingles, posts, fencing, boats.
Redwood	Durable in contact with soil. Weak and brittle, but very easy to work.	Ties, posts and poles, general construction.
Cypress	Very durable, light, close grained, easily worked, takes high polish	Shingles, poles, siding, interior finish, general building lumber.

TABLE V

Static Strength and Stiffness of Wood

The values given in this table are average values based on results of a large number of tests mainly by the U. S. Forest Service laboratories.

Wood	Weight (Dry) Lb. per Cu. Ft.	Strength and Stiffness (All Values in Lb. per Sq. In.)							
		Compression Parallel to Grain			Compression Across Grain	Shear Parallel to Grain	Flexure		
		Modulus of Elasticity	Proportional Elastic Limit	Ultimate	Proportional Elastic Limit	Ultimate	Modulus of Elasticity	Proportional Elastic Limit	Modulus of Rupture
Hardwood, small straight-grained test specimens, air dry (*a*)									
White Oak	50	.....	5,000	8,500	2,200	1,000	2,000,000	9,500	13,000
Red Oak	45	.....	4,500	7,000	2,000	1,000	1,950,000	9,000	11,500
Hickory	50	.....	5,500	10,000	2,500	1,200	2,400,000	11,500	16,000
Maple	43	.....	5,000	8,000	1,300	500	1,300,000	5,000	7,000
Soft wood, small straight-grained test specimens, air dry (*a*)									
Yellow Pine	34	1,450,000	4,000	5,800	900	800	1,650,000	6,500	11,000
Douglas Fir	28	1,100,000	3,500	5,000	800	500	1,700,000	6,500	10,500
Norway Pine and White Pine (*b*)	24	1,350,000	3,500	5,000	700	400	1,400,000	6,300	8,000
Hemlock	28	1,900,000	4,500	5,000	600	400	1,650,000	6,300	10,000
Tamarack	31	1,350,000	3,500	4,500	700	400	1,600,000	7,500	13,000
Redwood	22	950,000	3,800	5,000	550	350	1,150,000	4,500	7,500
Spruce	24	1,000,000	5,500	7,000	700	800	1,600,000	8,000	10,000
Cedar	23	900,000	4,000	5,000	700	400	900,000	5,500	6,500
Cypress	29	.....	.....	6,000	800	500	1,300,000	6,500	7,500
Soft wood, full-size structural timber with ordinary defects, green.									
Yellow Pine	34	1,100,000	2,800	3,500	440	350	1,350,000	3,800	4,500
Douglas Fir	28	1,150,000	2,500	4,500	430	300	1,550,000	3,900	4,300
Norway Pine White Pine (*c*)	24	1,350,000	2,000	2,800	350	200	1,400,000	2,800	4,000
Hemlock	28	2,000,000	3,000	4,000	300	200	1,700,000	2,800	4,300
Tamarack	31	1,300,000	2,200	3,000	300	200	1,300,000	2,500	3,500
Redwood	22	950,000	2,000	2,900	350	150	900,000	2,300	2,500
Spruce	24	1,000,000	2,300	3,000	400	200	1,500,000	2,500	3,500
Cedar	23	900,000	2,000	2,600	380	130	850,000	2,800	3,400

(*a*) For most small pieces of wood it is possible to use air-dry wood and to keep it in that condition. For cases where there is likelihood of reabsorption of water the values given will be diminished by about one-third. For the rare cases in which kiln-dry wood can be used and protected from reabsorbing water the values given will be increased by about one-third.

(*b*) The wood marketed as Western pine includes several kinds of wood. It has about the same average strength values as Norway pine.

TABLE VI

Compressive Strength of Piers

The values in this table are based on test data from Watertown Arsenal, the U. S. Bureau of Standards, Cornell University, and the University of Illinois.

Make-up of Pier	Ultimate Compressive Strength Lb. per Sq. In.
Terra-cotta block, Portland cement mortar	3000
Vitrified brick, Portland cement mortar	2800
Pressed brick, Portland cement mortar	2000
Pressed brick, lime mortar	1400
Common brick, Portland cement mortar	1000
Common brick, lime mortar	700
Sand-lime brick, lime mortar	500*

* Strength of sand-lime brick piers is estimated from comparative strength of individual sand-lime bricks and common bricks.

TABLE VII

Compressive Strength of Portland Cement Concrete

The values in this table are from the report of the Joint Committee on Concrete and Reinforced Concrete. The values are based on test data from specimens in the form of cylinders 8 inches diameter by 16 inches long, stored under laboratory conditions and tested when 28 days old. The fine aggregate except for gravel and cinders is sand, and enough was used to fill the voids in the coarse aggregate; usually the ratio (by volume) of fine aggregate to coarse aggregate was about 1:2.

Coarse Aggregate	Proportion of Cement to Aggregate				
	1 : 3	1 : 4.5	1 : 6	1 : 7.5	1 : 9
All values in lb. per sq. in.					
Granite, trap rock	3300	2800	2200	1800	1400
Gravel, hard limestone, hard sandstone	3000	2500	2200	1800	1400
Soft limestone, soft sandstone	2200	1800	1500	1200	1000
Cinders	800	700	600	500	400

TABLE VIII

Results of Shearing Tests of Portland Cement Concrete

The values in this table are summaries of results given in Bulletin No. 8 of the University of Illinois Engineering Experiment Station. It is probable that concrete, a brittle material, fails in tension on inclined planes rather than in shear. The values here given are the unit-stresses in shear at failure for specimens so tested that flexure, direct tension, and direct compression were minimized. The concrete was 60 days old when tested. The aggregate used was bank sand and soft limestone.

Proportion of Cement to Aggregate	Ultimate, Lb. per Sq. In.	
	Computed Unit-stress in Shear at failure; Shearing Tests	Compressive Strength as Given by Compression Tests
1 : 6	1290	2430
1 : 9	1090	1290

TABLE IX

Static Strength and Stiffness of American Building Stone

Values based on test data from Watertown Arsenal.

Stone	Weight (Av.) Lb. per Cu. Ft.	Ultimate in Compression Lb. per Sq. In.			Computed Ultimate in Shear Lb. per Sq. In. Av.	Modulus of Rupture (Flexure) Lb. per Sq. In.			Modulus of Elasticity (Flexure) Lb. per Sq. In.
		Max.	Min.	Av.		Max.	Min.	Av.	
Granite....	165	26,000	15,000	20,000	2300	2200	1200	1600	7,500,000
Marble.....	170	16,100	10,300	12,500	1300	2300	800	1500	8,200,000
Limestone..	160	20,000	3,200	9,000	1400	2700	250	1200	8,400,000
Sandstone..	135	19,000	6,700	12,500	1700	2200	500	1500	3,300,000
Slate.....	175	.....	.....	15,000	.....	.....	.....	8500	14,000,000

TABLE X

Working Stresses for Structural Materials Subjected to Static Loading

The values in this table are from the building ordinances of the city of Chicago.

Allowable Compressive Stresses for Masonry (lb. per sq. in.):

Coursed rubble, Portland cement mortar	200
Ordinary rubble, Portland cement mortar	100
Coursed rubble, lime mortar	120
Ordinary rubble, lime mortar	60
First-class granite masonry, Portland cement mortar	600
First-class limestone and sandstone masonry, Portland cement mortar	400
Portland cement concrete, 1–2–4 mixture, machine mixed	400
Portland cement concrete, 1–2½–5 mixture, machine mixed	350
Portland cement concrete, 1–3–6 mixture, machine mixed	300
Paving brick masonry, Portland cement mortar	350
Selected hard common brick masonry, Portland cement mortar	200
Common brick masonry, Portland cement mortar	175
Common brick masonry, lime mortar	100

Allowable Stresses for Timber (lb. per sq. in.):

Wood	Extreme Fibers in Beams	Compression Along the Grain*	Compression Across the Grain	Shear Along the Grain
Douglas fir and long leaf yellow pine	1300	1100	250	130
Oak	1200	900	500	200
Short leaf yellow pine	1000	800	250	120
Norway pine and white pine	800	700	200	80
Hemlock	600	500	150	60

* For columns these values are too high; a column formula must be used.

Allowable Stresses for Iron and Steel (lb. per sq. in.):

Kind of Stress	Rolled Steel	Steel Castings	Wrought Iron	Cast Iron
Tension on net section	16,000	16,000	12,000	.....
Max. compression on gross section	14,000	14,000	10,000	10,000
Bending on extreme fiber	16,000	16,000	12,000	.....
Bending on extreme fiber, tension	.....	.....	.....	3,000
Bending on extreme fiber, compression	.....	.....	.....	10,000
Bending on extreme fiber, pins	25,000	.....	.....	.....
Shear, pins and shop-driven rivets	12,000	.....	.....	.....
Shear, field-driven rivets	10,000	.....	.....	.....
Shear on rolled steel shapes	12,000	.....	.....	.....
Shear plate girder webs, gross section	10,000	.....	.....	.....
Shear on brackets	.....	.....	.....	2,000
Bearing, pins and shop-driven rivets	25,000	.....	.....	.....
Bearing, field-driven rivets	20,000	.....	.....	.....

APPENDIX III

PROPERTIES OF ROLLED STEEL SECTIONS

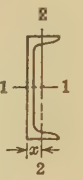
TABLE A

PROPERTIES OF I-BEAMS

Depth of Beam	Weight per Foot	Area of Section	Width of Flange	Thickness of Web	Axis 1–1			Axis 2–2		
					I	r	I/c	I	r	I/c
In.	Lb.	In.^{2}	In.	In.	In.^{4}	In.	In.^{3}	In.^{4}	In.	In.^{3}
27	90.0	26.34	9.000	0.524	2958.3	10.60	219.1	75.3	1.69	16.7
24	120.0	35.13	8.048	0.798	3010.8	9.26	250.9	84.9	1.56	21.1
24	79.9	23.33	7.000	0.500	2087.2	9.46	173.9	42.9	1.36	12.2
24	74.2	21.70	9.000	0.476	1950.1	9.48	162.5	61.2	1.68	13.6
21	60.4	17.68	8.250	0.428	1235.5	8.36	117.7	43.5	1.57	10.6
20	100.0	29.20	7.273	0.873	1648.3	7.51	164.8	52.4	1.34	14.4
20	65.4	19.08	6.250	0.500	1169.5	7.83	116.9	27.9	1.21	8.9
18	90.0	26.29	7.236	0.796	1256.5	6.91	139.6	51.9	1.40	14.3
18	54.7	15.94	6.000	0.460	795.5	7.07	88.4	21.2	1.15	7.1
18	48.2	14.09	7.500	0.380	737.1	7.23	81.9	30.0	1.46	8.0
15	75.0	21.85	6.278	0.868	687.2	5.61	91.6	30.6	1.18	9.8
15	42.9	12.49	5.500	0.410	441.8	5.95	58.9	14.6	1.08	5.3
15	37.3	10.91	6.750	0.332	405.5	6.10	54.1	19.9	1.35	5.9
12	55.0	16.04	5.600	0.810	319.3	4.46	53.2	17.3	1.04	6.2
12	40.8	11.84	5.250	0.460	268.9	4.77	44.8	13.8	1.08	5.3
12	31.8	9.26	5.000	0.350	215.8	4.83	36.0	9.5	1.01	3.8
12	27.9	8.15	6.000	0.284	199.4	4.95	33.2	12.6	1.24	4.2
10	40.0	11.69	5.091	0.741	158.0	3.68	31.6	9.4	0.90	3.7
10	30.0	8.75	4.797	0.447	133.5	3.91	26.7	7.6	0.93	3.2
10	25.4	7.38	4.660	0.310	122.1	4.07	24.4	6.9	0.97	3.0
10	22.4	6.54	5.500	0.252	113.6	4.17	22.7	9.0	1.17	3.3
9	35.0	10.22	4.764	0.724	111.3	3.30	24.7	7.3	0.84	3.0
9	30.0	8.76	4.601	0.561	101.4	3.40	22.5	6.4	0.85	2.8
9	21.8	6.32	4.330	0.290	84.9	3.67	18.9	5.2	0.90	2.4
8	25.5	7.43	4.262	0.532	68.1	3.03	17.0	4.7	0.80	2.2
8	18.4	5.34	4.000	0.270	56.9	3.26	14.2	3.8	0.84	1.9
7	20.0	5.83	3.860	0.450	41.9	2.68	12.0	3.1	0.74	1.6
7	15.3	4.43	3.660	0.250	36.2	2.86	10 4	2.7	0.78	1.5
6	17.25	5.02	3.565	0.465	26.0	2.28	8.7	2.3	0.68	1.3
6	12.5	3.61	3.330	0.230	21.8	2.46	7.3	1.8	0.72	1.1
5	10.0	2.87	3.000	0.210	12.1	2.05	4.8	1.2	0.65	0.82
4	10.5	3.05	2.870	0.400	7.1	1.52	3.5	1.0	0.57	0.70
3	7.5	2.17	2.509	0.349	2.9	1.15	1.9	0.59	0.52	0.47
3	5.7	1.64	2.330	0.170	2.5	1.23	1.7	0.46	0.53	0.40

TABLE B

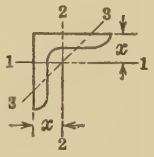
PROPERTIES OF CHANNELS



Depth of Channel	Weight per Foot	Area of Section	Width of Flange	Thickness of Web	Axis 1–1			Axis 2–2			
					I	r	I/c	I	r	I/c	x
In.	Lb.	In.^{2}	In.	In.	In.^{4}	In.	In.^{3}	In.^{4}	In.	In.^{3}	In.
15	55.0	16.11	3.814	0.814	429.0	5.16	57.2	12.1	0.87	4.1	0.82
15	45.0	13.17	3.618	0.618	373.9	5.33	49.8	10.3	0.88	3.6	0.79
15	33.9	9.90	3.400	0.400	312.6	5.62	41.7	8.2	0.91	3.2	0.79
12	40.0	11.73	3.415	0.755	196.5	4.09	32.8	6.6	0.75	2.5	0.72
12	20.7	6.03	2.940	0.280	128.1	4.61	21.4	3.9	0.81	1.7	0.70
10	35.0	10.27	3.180	0.820	115.2	3.34	23.0	4.6	0.67	1.9	0.69
10	15.3	4.47	2.600	0.240	66.9	3.87	13.4	2.3	0.72	1.2	0.64
9	25.0	7.33	2.812	0.612	70.5	3.10	15.7	3.0	0.64	1.4	0.61
9	13.4	3.89	2.430	0.230	47.3	3.49	10.5	1.8	0.67	0.97	0.61
8	21.25	6.23	2.619	0.579	47.6	2.77	11.9	2.2	0.60	1.1	0.59
8	11.5	3.36	2.260	0.220	32.3	3.10	8.1	1.3	0.63	0.79	0.58
7	19.75	5.79	2.509	0.629	33.1	2.39	9.4	1.8	0.56	0.96	0.58
7	9.8	2.85	2.090	0.210	21.1	2.72	6.0	0.98	0.59	0.63	0.55
6	15.5	4.54	2.279	0.559	19.5	2.07	6.5	1.3	0.53	0.73	0.55
6	8.2	2.39	1.920	0.200	13.0	2.34	4.3	0.70	0.54	0.50	0.52
5	11.5	3.36	2.032	0.472	10.4	1.76	4.1	0.82	0.49	0.54	0.51
5	6.7	1.95	1.750	0.190	7.4	1.95	3.0	0.48	0.50	0.38	0.49
4	7.25	2.12	1.720	0.320	4.5	1.47	2.3	0.44	0.46	0.35	0.46
4	5.4	1.56	1.580	0.180	3.8	1.56	1.9	0.32	0.45	0.29	0.46
3	6.0	1.75	1.596	0.356	2.1	1.08	1.4	0.31	0.42	0.27	0.46
3	4.1	1.19	1.410	0.170	1.6	1.17	1.1	0.20	0.41	0.21	0.44

TABLE C

PROPERTIES OF EQUAL ANGLES

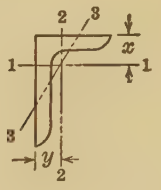


Size	Weight per Foot	Area of Section	Axis 1–1 and Axis 2–2				Axis 3–3*
			I	r	I/c	x	r
Inches	Lb.	In.^{2}	In.^{4}	In.	In.^{3}	In.	In.
8 ×8 ×1⅛	56.9	16.73	98.0	2.42	17.5	2.41	1.55
8 ×8 × ½	26.4	7.75	48.6	2.51	8.4	2.19	1.58
6 ×6 ×1	37.4	11.00	35.5	1.80	8.6	1.86	1.16
6 ×6 × ⅜	14.9	4.36	15.4	1.88	3.5	1.64	1.19
5 ×5 ×1	30.6	9.00	19.6	1.48	5.8	1.61	0.96
5 ×5 × ⅝	12.3	3.61	8.7	1.56	2.4	1.39	0.99
4 ×4 ×13/16	19.9	5.84	8.1	1.18	3.0	1.29	0.77
4 ×4 × ¼	6.6	1.94	3.0	1.25	1.0	1.09	0.79
3½×3½×13/16	17.1	5.03	5.3	1.02	2.3	1.17	0.67
3½×3½× ¼	5.8	1.69	2.0	1.09	0.79	0.97	0.69
3 ×3 × ⅝	11.5	3.36	2.6	0.88	1.3	0.98	0.57
3 ×3 × ½	9.4	2.75	2.2	0.90	1.1	0.93	0.58
3 ×3 × ⅜	7.2	2.11	1.8	0.91	0.83	0.89	0.58
3 ×3 × ¼	4.9	1.44	1.2	0.93	0.58	0.84	0.59
2½×2½× ½	7.7	2.25	1.2	0.74	0.73	0.81	0.47
2½×2½× ⅜	2.08	0.61	0.38	0.79	0.20	0.67	0.50
2 ×2 × 7/16	5.3	1.56	0.54	0.59	0.40	0.66	0.39
2 ×2 × ⅛	1.65	0.48	0.19	0.63	0.13	0.55	0.40
1½×1½× ⅜	3.35	0.98	0.19	0.44	0.19	0.51	0.29
1½×1½× ⅛	1.23	0.36	0.08	0.46	0.07	0.42	0.30
1 ×1 × ¼	1.49	0.44	0.04	0.29	0.06	0.34	0.19
1 ×1 × ⅛	0.80	0.23	0.02	0.31	0.03	0.30	0.19

* Axis 3–3 is the principal axis about which the radius of gyration is least.

TABLE D

PROPERTIES OF UNEQUAL ANGLES



Size	Weight per Foot	Area of Section	Axis 1–1				Axis 2–2				Axis 3–3*
			I	r	I/c	x	I	r	I/c	y	r
Inches	Lb.	In.^{2}	In.^{4}	In.	In.^{3}	In.	In.^{4}	In.	In.^{3}	In.	In.
8 ×6 ×1	44.2	13.00	80.8	2.49	15.1	2.65	38.8	1.73	8.9	1.65	1.28
8 ×6 × 7/16	20.2	5.93	39.2	2.57	7.1	2.45	19.3	1.80	4.2	1.45	1.30
8 ×3½×1	35.7	10.50	66.2	2.51	13.7	3.17	7.8	0.86	3.0	0.92	0.73
8 ×3½× 7/16	16.5	4.84	32.5	2.59	6.4	2.95	4.1	0.92	1.5	0.70	0.74
7 ×3½×1	32.3	9.50	45.4	2.19	10.6	2.70	7.5	0.89	3.0	0.96	0.74
7 ×3½× 3/8	13.0	3.80	19.6	2.27	4.3	2.48	3.5	0.96	1.3	0.73	0.76
6 ×4 ×1	30.6	9.00	30.8	1.85	8.0	2.17	10.8	1.09	3.8	1.17	0.85
6 ×4 × 3/8	12.3	3.61	13.5	1.93	3.3	1.94	4.9	1.17	1.6	0.94	0.88
6 ×3½×1	28.9	8.50	29.2	1.85	7.8	2.26	7.2	0.92	2.9	1.01	0.74
6 ×3½× 5/16	9.8	2.87	10.9	1.95	2.7	2.01	2.9	1.00	1.0	0.76	0.77
5 ×4 × 7/8	24.2	7.11	16.4	1.52	5.0	1.71	9.2	1.14	3.3	1.21	0.84
5 ×4 × 3/8	11.0	3.23	8.1	1.59	2.3	1.53	4.7	1.20	1.6	1.03	0.86
5 ×3½× 7/8	22.7	6.67	15.7	1.53	4.9	1.79	6.2	0.96	2.5	1.04	0.75
5 ×3½× 5/16	8.7	2.56	6.6	1.61	1.9	1.59	2.7	1.03	1.0	0.84	0.76
5 ×3½× ½	13.6	4.00	10.0	1.58	3.0	1.66	4.0	1.01	1.6	0.91	0.75
5 ×3 ×13/16	19.9	5.84	14.0	1.55	4.5	1.86	3.7	0.80	1.7	0.86	0.64
5 ×3 × 5/16	8.2	2.40	6.3	1.61	1.9	1.68	1.8	0.85	0.75	0.68	0.66
4½×3 ×13/16	18.5	5.43	10.3	1.38	3.6	1.65	3.6	0.81	1.7	0.90	0.64
4½×3 × 5/16	7.7	2.25	4.7	1.44	1.5	1.47	1.7	0.87	0.75	0.72	0.66
4 ×3½×13/16	18.5	5.43	7.8	1.19	2.9	1.36	5.5	1.01	2.3	1.11	0.72
4 ×3½× 5/16	7.7	2.25	3.6	1.26	1.3	1.18	2.6	1.07	1.0	0.93	0.73
4 ×3 ×13/16	17.1	5.03	7.3	1.21	2.9	1.44	3.5	0.83	1.7	0.94	0.64
4 ×3 × ¼	5.8	1.69	2.8	1.28	1.0	1.24	1.4	0.89	0.60	0.74	0.65
3½×3 ×13/16	15.8	4.62	5.0	1.04	2.2	1.23	3.3	0.85	1.7	0.98	0.62
3½×3 × ¼	5.4	1.56	1.9	1.11	0.78	1.04	1.3	0.91	0.58	0.79	0.63
3 ×2½× 9/16	9.5	2.78	2.3	0.91	1.2	1.02	1.4	0.72	0.82	0.77	0.52
3 ×2½× ¼	4.5	1.31	1.2	0.95	0.56	0.91	0.74	0.75	0.40	0.66	0.53
3 ×2 × ½	7.7	2.25	1.9	0.92	1.0	1.08	0.67	0.55	0.47	0.58	0.43
3 ×2 × ¼	4.1	1.19	1.1	0.95	0.54	0.99	0.39	0.57	0.25	0.49	0.43
2½×2 × ½	6.8	2.00	1.1	0.75	0.70	0.88	0.64	0.56	0.46	0.63	0.42
2½×2 × ⅛	1.86	0.55	0.35	0.80	0.20	0.74	0.20	0.61	0.13	0.49	0.43
2 ×1½× 3/8	3.99	1.17	0.43	0.61	0.34	0.71	0.21	0.42	0.20	0.46	0.32
2 ×1½× ⅛	1.44	0.42	0.17	0.64	0.13	0.62	0.09	0.45	0.08	0.37	0.33

* Axis 3–3 is the principal axis about which the radius of gyration is least.

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